# Characterizations of Mixed Binary Convex Quadratic Representable Sets

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March 28, 2017

**Abstract**

Representability results play a fundamental role in optimization since they provide characterizations of the feasible sets that arise from optimization problems. In this paper we study the sets that appear in the feasibility version of mixed binary convex quadratic optimization problems. We provide a complete characterization of the sets that can be obtained as the projection of such feasible regions in a higher dimensional space. In addition, we provide a complete characterization of these sets in the special cases where (i) the feasible region is bounded, (ii) only binary extended variables are present, and (iii) only continuous variables are present.

# 1 Introduction

## 1.1 Background

The theory of representability studies one fundamental question: Given a system of algebraic constraints of a specified form, which subsets of $\mathbb{R}^n$ can be represented in this way, possibly using additional variables? A thorough answer to this question would be given by a complete characterization of representable sets. Complete characterizations are useful in that they demonstrate the class of problems which can be modeled using a fixed set of constraints.

Representability is well understood for systems of linear inequalities. It is well known that the projection of a set described by finitely many linear inequalities is again described by finitely many linear inequalities. It follows from the Minkowski-Weyl Theorem that such sets decompose as the Minkowski sum of a polytope and a polyhedral cone.

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In the case of mixed binary linear systems, i.e., using only linear constraints and allowing both continuous and binary extended variables, a geometric characterizations has been given by Jeroslow [9]. He shows that the sets that can be represented with these constraints are precisely the sets that decompose as a finite union of polyhedra which share a common recession cone.

In [10], Jeroslow and Lowe consider rational linear inequalities and extended variables that can be both continuous and integer. They show that the the sets that can be represented using these constraints are precisely the sets that can be realized as a Minkowski sum of a finite union of rational polytopes and an integer cone of a finite number of integral vectors.

Representability has also been studied in the case of nonlinear constraints, but few complete characterizations have been established. In second-order cone programming a linear functional is maximized over a set defined by linear inequalities and constraints of the form $||Ax + b||_2 \leq c^\top x + d$. These constraints are quite general and can express a variety of different constraints, including convex quadratic inequalities. There has been a large amount of work [11, 12] that shows different second order cone formulations for a wide range of problems. However, a complete characterization similar to the ones obtained by Jeroslow and Lowe is missing.

In semidefinite programming, a linear functional is maximized over a set defined by a linear matrix inequality, i.e., a constraint of the form $A_0 + \sum_{i=1}^n x_i A_i \succeq 0$ where the $A_i$ are symmetric matrices. A linear matrix inequality defines a closed, convex, semialgebraic set known as a spectrahedon. In [8], Helton and Vinnikov introduce the notion of rigid convexity and conjecture that a set is a spectrahedron if and only if it is rigidly convex. Another conjecture is stated in [7] where Helton and Nie study which sets can be represented as the projection of a spectrahedon in a higher dimensional space. They conjecture that every convex semialgebraic set can be represented as the projection of a spectrahedron.

The difficulty in establishing these conjectures, as well as forming a characterization in the case of second-order cone programming, lies in the complexity of describing the projection of semialgebraic sets.

In hopes of bridging the gap between characterization results for linear systems and similar results for nonlinear systems, we have considered in [5, 4] sets described by linear inequalities and a single convex quadratic inequality. We observed that a characterization of sets representable by more than one convex quadratic inequality seems to be currently out of reach. In fact, the intersection of two convex quadratic inequalities in $\mathbb{R}^3$ may project to a semialgebraic set described by polynomials of degree four in $\mathbb{R}^2$.

In [5], we characterized sets described using linear constraints, a single convex quadratic inequality, and allowing for both continuous and binary extended variables, under the additional assumption that the quadratic inequality could be factored as $(w - c)^\top Q(w - c) \leq \gamma$. We call such inequalities *ellipsoidal* and similarly define *ellipsoidal regions* to be the regions they describe. We show that a set $S \subseteq \mathbb{R}^n$ can be represented with these constraints if and only if there exist ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}.$$

In [4], we consider the same systems but with rational data and extended variables that can be both continuous and integer. We show that a set $S \subseteq \mathbb{R}^n$ can be represented with these constraints if and only if there exist rational ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, rational polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and integral vectors $r^j \in \mathbb{Z}^n$, $j = 1, \ldots, t$ such that

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \text{int.cone}\{r^1, \ldots, r^t\},$$

where int.cone denotes the integer cone. We note that both of these results are direct extensions of the characterizations obtained by Jeroslow and Lowe in the case of linear systems.

## 1.2 Our Contributions

In this paper, we consider sets described by linear inequalities and a single convex quadratic inequality. We say that a region $\mathcal{Q}$ is a *convex quadratic region in* $\mathbb{R}^n$ if

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Q x + h^\top x + g \leq 0\}$$

for a positive semidefinite matrix $Q \in \mathbb{R}^{n \times n}$, a vector $h \in \mathbb{R}^n$, and $g \in \mathbb{R}$. In general, a convex quadratic inequality cannot be factored into an ellipsoidal inequality. This implies that the family of ellipsoidal regions is a strict subset of the family of convex quadratic regions.

We say that a set $S \subseteq \mathbb{R}^n$ is *mixed binary convex quadratic representable* if it can be obtained as the projection onto $\mathbb{R}^n$ of the solution set of a system of the form

$$\begin{aligned} & Dw \leq d \\ & w^\top Q w + h^\top w + g \leq 0 \\ & w \in \mathbb{R}^{n+p} \times \{0,1\}^q, \end{aligned} \tag{1}$$

where $Q$ is positive semidefinite. Note that if a set $S$ is the projection of the solution set of a system of the form (1), but with bounded integer variables in the place of the binary variables, then $S$ is also the projection of the solution set of a system of the form (1). We also note that since any convex quadratic region is second-order cone representable, the sets that we characterize can be represented with second-order cone constraints and mixed binary extended variables.

There is a strong connection between mixed binary convex quadratic representable sets and mixed binary convex quadratic programming (MBCQP). This class of problems has applications in many areas, including portfolio optimization and machine learning [2, 1]. Since optimal solutions of MBCQP problems have polynomial size (see [13, 3]), any MBCQP is polynomially equivalent to a polynomial number of MBCQP feasibility problems. In particular, each feasibility problem is over a set of the form (1). Moreover, by linearizing the objective, any MBCQP can be transformed to the problem of minimizing a linear function over a set described by (1).

In this paper, we present characterization results for a number of cases of mixed binary convex quadratic representable sets. See Figure 1 and Figure 2 for examples of representable sets. Before proceeding with the proofs, we provide a brief description of the statements and preview the proof techniques.

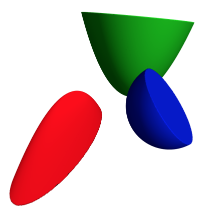

Figure 1: A bounded mixed binary convex quadratic representable set

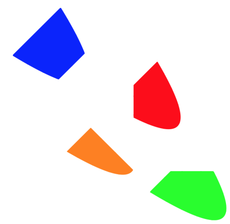

Figure 2: A binary convex quadratic representable set

In Section 2, we characterize sets that are *bounded mixed binary convex quadratic representable*, defined as the projection of the solution set of (1) where $Dw \le d$ describes a polytope.

**Theorem 1.** *A set $S \subseteq \mathbb{R}^n$ is bounded mixed binary convex quadratic representable if and only if there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, such that*

$$S = \bigcup_{i=1}^{k} (\mathcal{Q}_i \cap \mathcal{P}_i). \tag{2}$$

The characterization given in Theorem 1 is quite general in the sense that there is no restriction on the structure of the convex quadratic regions that may appear in the union (2). This is quite similar to what holds for ellipsoidal regions in [5], and indeed the fact that each region $\mathcal{Q}_i \cap \mathcal{P}_i$ in (2) is bounded allows us to find an extended formulation where each $\mathcal{Q}_i$ appears as a binary slice of a global convex quadratic region $\mathcal{Q}$. In the case of ellipsoidal regions this level of generality still holds even for unbounded regions. We will see in Section 5, where the bounded assumption is removed, that although a decomposition of representable sets into a union (2) holds, the convex quadratic regions that appear must share common structure.

In Section 3, we characterize sets that are *binary convex quadratic representable*, i.e., where $p = 0$ is fixed in (1). In order to provide a characterization of such sets, we need to remark on the geometry of convex quadratic sets in more detail. We make the following observation and definition. Let $\mathcal{Q} \subseteq \mathbb{R}^n$ be a convex quadratic region defined by

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Q x + h^\top x + g \leq 0\},$$

where $Q \succeq 0$. Since $Q$ is symmetric, it is a fact of linear algebra that $\mathbb{R}^n = \text{range}(Q) \oplus \ker(Q)$. Thus, we can decompose $h = Qw + v$ where $v \in \ker(Q)$ is uniquely determined. We note that $\mathcal{Q}$ is an ellipsoidal region if and only if $v = 0$.

The pair $Q, v$ defining $\mathcal{Q}$ is essential in understanding the geometry of $\mathcal{Q}$. In this vein, we say that two convex quadratic regions $\mathcal{Q}_1, \mathcal{Q}_2 \subseteq \mathbb{R}^n$ have the *same shape* if there exists a positive semidefinite matrix $Q$, a vector $v \in \ker(Q)$, vectors $w^i$, and scalars $g_i$ such that

$$\mathcal{Q}_i = \{x \in \mathbb{R}^n \mid x^\top Q x + (Qw^i + v)^\top x + g_i \leq 0\}, \quad i = 1, 2.$$

Geometrically, this means that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have the same structure, up to translation and constant term. Note that having the same shape is a transitive property and thus it makes sense for a collection of convex quadratic regions to have the same shape. We will establish the following result.

**Theorem 2.** *A set $S \subseteq \mathbb{R}^n$ is binary convex quadratic representable if and only if there exist convex quadratic regions $\mathcal{Q}_1, \ldots, \mathcal{Q}_k \subseteq \mathbb{R}^n$ with the same shape, and polyhedra $\mathcal{P}_1, \ldots, \mathcal{P}_k \subseteq \mathbb{R}^n$ with the same recession cone such that*

$$S = \bigcup_{i=1}^{k} (\mathcal{Q}_i \cap \mathcal{P}_i). \tag{3}$$

In Section 4 we obtain an algebraic characterization of *continuous convex quadratic representable sets*, i.e., where $q = 0$ is fixed in (1). This sort of algebraic description is quite different from the geometric characterizations obtained prior to this. The combination of extended continuous variables and unbounded regions creates a number of difficulties. Part of this difficulty is due to the complexity of describing the projection of semialgebraic sets. While methods such as Cylindrical Algebraic Decomposition may be used to compute the projection of (1), these outputs give little insight into the requirements that must be met for a set to be representable. Another difficulty is that we are not able to use standard disjunctive extended formulations. This is due to the fact that in general a convex quadratic region cannot be decomposed as the Minkowski sum of a bounded region and a polyhedral cone, in contrast to both polyhedra and ellipsoidal regions (see [5]).

In order to overcome these difficulties, we design a method to explicitly compute $S := \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ for a general convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ and a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p}$. A crucial step, stated in Proposition 1, is the construction of a 'shadowing skeleton' of $\mathcal{Q} \cap \mathcal{P}$, namely a finite set $\mathcal{L}$ of $n$-dimensional affine spaces that satisfy $\text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \bigcup_{L \in \mathcal{L}} \text{proj}_n(\mathcal{Q} \cap L) \cap$

$\text{proj}_n(\mathcal{P} \cap L)$. The idea of this skeleton is a general version of the projection method done in [5]. The explicit computation of $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ leaves us with an algebraic description of $S$ which we use to make a technical definition of sets of *Type 1* and *Type 2*. These definitions allow us to characterize continuous convex quadratic representable sets as shown in Theorem 3.

In Section 5 we present an algebraic characterization for the general case of mixed binary convex quadratic representable sets. Theorem 4 follows naturally from the combination of our results in Section 2 and Section 4. It follows immediately from Theorem 4 that mixed binary convex quadratic representable sets can be expressed as a finite union of $\mathcal{Q}_i \cap \mathcal{P}_i$ for convex quadratic regions $\mathcal{Q}_i$ and polyhedron $\mathcal{P}_i$. However, in contrast to Theorem 1, the convex quadratic regions $\mathcal{Q}_i$ that appear in a decomposition share a common geometry. This compatibility requirement is captured by our definition of sets with the *same structure* which follows from combining our definition of sets of Type 1 and Type 2 with the notion of convex quadratic regions with the same shape.

In Section 6 we work towards a geometric characterization of mixed binary convex quadratic representable sets. We derive obvious necessary conditions for a set to be representable from Theorem 3 and Theorem 4, and explore whether these necessary conditions are in fact are sufficient. We conclude the section, and paper, with open questions, and an instructive example of a set that is not mixed binary convex quadratic representable.

### 1.3 Notation

In this work, we will use the following notation. Given a set $E \subseteq \mathbb{R}^n \times \mathbb{R}^p$ and a vector $\bar{y} \in \mathbb{R}^p$, we define the *$\bar{y}$-restriction* of $E$ as $E|_{y=\bar{y}} = \{x \in \mathbb{R}^n \mid (x, \bar{y}) \in E\}$. Note that $E|_{y=\bar{y}}$ geometrically consists of the intersection of $E$ with coordinate hyperplanes. We write $\text{proj}_n(E)$ for the orthogonal projection of $E$ onto the space $\mathbb{R}^n$. We denote by $\text{rec}(E)$ the recession cone of $E$ and by $\text{lin}(E)$ the lineality space of $E$.

Given a matrix $A$ we denote by $\text{range}(A)$ the range of $A$ and by $\ker(A)$ the kernel of $A$. If $A$ is positive semidefinite, we write $A \succeq 0$. This implies that $A$ is symmetric. Given a half-space $H^+ = \{x \in \mathbb{R}^n \mid a^\top x \leq b\}$, we write $H$ for the hyperplane $\{x \in \mathbb{R}^n \mid a^\top x = b\}$.

## 2 The Bounded Case

In this section we give a characterization of bounded mixed binary convex quadratic representable sets, i.e., when the system $Dw \leq d$ in (1) describes a polytope.

*Proof of Theorem 1.* We prove sufficiency of the condition by giving an explicit extended formulation for a set $S$ described by (2). It is similar to the proof of Theorem 1 in [5]. Assume that we are given a set $S$ as in (2), where $\mathcal{Q}_i = \{x \in \mathbb{R}^n \mid x^\top Q_i x + (h^i)^\top x + g_i \leq 0\}$ are convex quadratic regions and $\mathcal{P}_i =$

$\{x \in \mathbb{R}^n \mid A^i x \leq b_i\}$ are polytopes. We now introduce new continuous variables $x^i \in \mathbb{R}^n$ and binary variables $\delta_i \in \{0,1\}$, for $i = 1, \ldots, k$, that will model the individual regions $\mathcal{Q}_i \cap \mathcal{P}_i$. Then $S$ can be described as the set of $x \in \mathbb{R}^n$ such that

$$\begin{aligned}
& x = \sum_{i=1}^{k} x^i \\
& A^i x^i \leq \delta_i b_i && i = 1, \ldots, k \\
& \sum_{i=1}^{k} \delta_i = 1 \\
& \sum_{i=1}^{k} \Big( (x^i)^\top Q_i x^i + (h^i)^\top x^i + \delta_i g_i \Big) \leq 0 \\
& 0 \leq \delta_i \leq 1 && i = 1, \ldots, k \\
& x^i \in \mathbb{R}^n, \ \ \delta_i \in \{0,1\} && i = 1, \ldots, k.
\end{aligned}$$

Now if $\delta_1 = 1$ the remaining $\delta_i$ must be 0. Then for each $x^i$ with $i \neq 1$, we have the constraint $A^i x^i \leq 0$ which has the single feasible point $x^i = 0$ since $\mathcal{P}_i$ is a polytope. The remaining constraints reduce to

$$\begin{aligned}
& x = x^1 \\
& A^1 x^1 \leq b_1 \\
& (x^1)^\top Q_1 x^1 + (h^1)^\top x^1 + g_1 \leq 0 \\
& x^1 \in \mathbb{R}^n.
\end{aligned}$$

This describes the set $\mathcal{Q}_1 \cap \mathcal{P}_1$ exactly. The remaining regions follow symmetrically.

We note that the constraint $\sum_{i=1}^{k} \Big( (x^i)^\top Q_i x^i + (h^i)^\top x^i + \delta_i g_i \Big) \leq 0$ describes a convex quadratic region since it can be described as a quadratic inequality with defining matrix a block diagonal matrix with blocks either 0 or $Q_i$, and each $Q_i \succeq 0$.

We show that the linear system is bounded by demonstrating that its set of feasible points is the graph of a linear transformation restricted to a polytope. Each system $A^i x^i \leq \delta_i b_i, \ 0 \leq \delta_i \leq 1$ is independent of any other $x^j, \delta_j$. Moreover, each system is bounded in $(x^i, \delta_i)$ as it is the convex hull of the polytope $\{x^i \in \mathbb{R}^n \ | A^i x^i \leq b_i\} \times \{1\}$ and the origin. Then the set of feasible points in $x^1, \ldots, x^k, \delta_1, \ldots, \delta_k$ is just a Cartesian product of bounded sets. Finally, the set of points $x$ satisfying equation $x = \sum_{i=1}^{k} x^i$ is bounded since it is the image of this Cartesian product under a linear transformation. Thus, $S$ is bounded mixed binary convex quadratic representable.

The remainder of the proof is devoted to proving necessity of the condition. We are given a convex quadratic region $\mathcal{Q}$ and a polytope $\mathcal{P}$ in $\mathbb{R}^{n+p+q}$, and define

$$\bar{S} := \mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0,1\}^q),$$
$$S := \text{proj}_n(\bar{S}).$$

We must show the existence of convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, such that

$$S = \bigcup_{i=1}^{k} (\mathcal{Q}_i \cap \mathcal{P}_i).$$

**Claim 1.** *It is enough to prove the theorem in the case $q = 0$.*

*Proof of claim.* Note that, using restrictions, we can write the set $S$ in the form

$$S = \bigcup_{\bar{z} \in \{0,1\}^q} \text{proj}_n(\bar{S}|_{z=\bar{z}}).$$

We first show that each restriction $\bar{S}|_{z=\bar{z}}$ can be written as $\mathcal{Q}' \cap \mathcal{P}'$ for some convex quadratic region $\mathcal{Q}' \subseteq \mathbb{R}^{n+p}$ and polytope $\mathcal{P}' \subseteq \mathbb{R}^{n+p}$. Let $\bar{z} \in \{0,1\}^q$. We note $\bar{S}|_{z=\bar{z}} = \mathcal{Q}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}}$. A short algebraic verification shows that $\mathcal{Q}' := \mathcal{Q}|_{z=\bar{z}}$ is a convex quadratic region and $\mathcal{P}' := \mathcal{P}|_{z=\bar{z}}$ is a polytope.

Now assuming the theorem in the case $q = 0$, for each $\bar{z} \in \{0,1\}^q$ we have $\text{proj}_n(\bar{S}|_{z=\bar{z}}) = \cup_{i=1}^{t}(\mathcal{Q}_i \cap \mathcal{P}_i)$. Since $S$ is the finite union of such sets, the theorem follows. $\diamond$

**Claim 2.** *It is enough to prove the theorem in the case $p = 1$.*

*Proof of claim.* Let $\mathcal{Q} \cap \mathcal{P} \subseteq \mathbb{R}^{n+p}$. We prove $S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ has the desired decomposition by induction on $p$. For this claim, we assume the base case, $p = 1$. Now let $p = k$, and suppose the statement holds for $p < k$. Given $\mathcal{Q} \cap \mathcal{P} \subseteq \mathbb{R}^{n+k}$, by the base case $p = 1$ we have

$$\text{proj}_{n+k-1}(\mathcal{Q} \cap \mathcal{P}) = \bigcup_{i=1}^{t} (\mathcal{Q}_i \cap \mathcal{P}_i)$$

where each $\mathcal{Q}_i$ is a convex quadratic region in $\mathbb{R}^{n+k-1}$ and each $\mathcal{P}_i$ is a polytope in $\mathbb{R}^{n+k-1}$. Since the projection of a union is the union of projections we have

$$S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \bigcup_{i=1}^{t} \text{proj}_n(\mathcal{Q}_i \cap \mathcal{P}_i).$$

Then by induction hypothesis, we have

$$S = \bigcup_{i=1}^{t} \Big( \bigcup_{j=1}^{s_i} (\mathcal{Q}_{i,j} \cap \mathcal{P}_{i,j}) \Big)$$

where each $\mathcal{Q}_{i,j}$ is a convex quadratic region in $\mathbb{R}^n$ and each $\mathcal{P}_{i,j}$ is a polytope in $\mathbb{R}^n$. $\diamond$

It remains to prove Theorem 1 in the case that we have a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+1}$ and a polytope $\mathcal{P} \subseteq \mathbb{R}^{n+1}$. The following claim then completes the proof of Theorem 1.

**Claim 3.** *Let $\mathcal{Q} \subseteq \mathbb{R}^{n+1}$ be a convex quadratic region described by*

$$\mathcal{Q} = \left\{ (x,y) \in \mathbb{R}^{n+1} \;\middle|\; \begin{pmatrix} x \\ y \end{pmatrix}^{\top} \begin{pmatrix} Q & q \\ q^{\top} & \gamma \end{pmatrix} + \begin{pmatrix} h_x \\ h_y \end{pmatrix}^{\top} \begin{pmatrix} x \\ y \end{pmatrix} + g \leq 0 \right\}$$

*and $\mathcal{P} \subseteq \mathbb{R}^{n+1}$ be a polytope. Then there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, such that* (2) *holds.*

*Proof of claim.* We first claim that $\text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \cup_{H \in \mathcal{H}} \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap H)$ where $\mathcal{H}$ is a finite set of hyperplanes. Suppose $\mathcal{P}$ is defined as the intersection $H_1^+ \cap \dots \cap H_s^+$ for half-spaces $H_i^+$. Let $\mathcal{H}$ be the subset of hyperplanes $H \in \{H_1, \dots, H_s\}$ such that $e_{n+1} \notin \text{lin}(H)$. In the case that $\gamma \neq 0$, define the hyperplane $H_0 := \{(x,y) \in \mathbb{R}^{n+1} \mid q^{\top} x + \gamma y = -\frac{1}{2} h_y\}$ and include $H_0$ in the set $\mathcal{H}$. This hyperplane has the property that for any fixed $\bar{x} \in \mathbb{R}^n$, the unique point $(\bar{x}, \bar{y}) \in H_0$ minimizes the univariate quadratic polynomial $q(\bar{x}, y)$ defining $\mathcal{Q}|_{x=\bar{x}}$. Moreover, $e_{n+1} \notin \text{lin}(H_0)$.

We claim that $\text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \cup_{H \in \mathcal{H}} \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap H)$. Let $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$. Define $L_{\bar{x}} = \{(\bar{x}, y) \in \mathbb{R}^{n+1} \mid (\bar{x}, y) \in \mathcal{Q} \cap \mathcal{P}\}$. Since $\mathcal{P}$ is a polytope, $L_{\bar{x}}$ is a non-empty line segment. Consider the endpoints, possibly both the same point, of $L_{\bar{x}}$. If either endpoint lies on the boundary of $\mathcal{P}$ then we are done as this point must lie on some $H \in \mathcal{H}$. Otherwise, both endpoints lie on the boundary of $\mathcal{Q}$ and are thus roots of the quadratic polynomial $q(\bar{x}, y)$. Then the midpoint of $L_{\bar{x}}$ lies on $H_0$.

It remains to show that for each $H \in \mathcal{H}$, there exists a convex quadratic region $\mathcal{Q}_H$ and a polytope $\mathcal{P}_H$ such that $\text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap H) = \mathcal{Q}_H \cap \mathcal{P}_H$. Let $H = \{(x,y) \in \mathbb{R}^{n+1} \mid a^{\top} x + \alpha y = b\}$ and note that $\alpha \neq 0$ since $e_{n+1} \notin \text{lin}(H)$. It follows that $\text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap H) = \text{proj}_n(\mathcal{Q} \cap H) \cap \text{proj}_n(\mathcal{P} \cap H)$ as there is a unique point $(x,y) \in H$ lying over any $x \in \mathbb{R}^n$.

We now show that $\mathcal{Q}_H := \text{proj}_n(\mathcal{Q} \cap H)$ is a convex quadratic region and $\mathcal{P}_H := \text{proj}_n(\mathcal{P} \cap H)$ is a polytope. The polyhedron $\mathcal{P}_H$ is clearly a polytope since $\mathcal{P}$ is a polytope. Define the invertible linear transformation $T_A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by multiplication on the left by the matrix

$$A = \begin{pmatrix} I_n & 0 \\ a^{\top} & \alpha \end{pmatrix}.$$

Then we have that

$$\mathcal{Q}_H = \text{proj}_n(\mathcal{Q} \cap H) = T_A(\mathcal{Q})|_{y=b}.$$

Note that $\mathcal{Q}_H$ is a convex quadratic region as it obtained from $\mathcal{Q}$ by an invertible linear transformation followed by fixing a single variable. $\diamond$

# 3 The Binary Case

In this section we characterize binary convex quadratic representable sets, i.e., when $p = 0$ is fixed in (1). We refer the reader back to the introduction for the definition of convex quadratic regions with the same shape. Before proving Theorem 2 we state a number of lemmas that detail the interaction of binary variables and convex quadratic regions.

**Lemma 1.** *Let $\mathcal{Q} \subseteq \mathbb{R}^{n+q}$ be a convex quadratic region. Then for all $\bar{z} \in \{0,1\}^q$, the sets $\mathcal{Q}|_{z=\bar{z}}$ are convex quadratic regions with the same shape.*

*Proof.* Assume that $\mathcal{Q}$ is given by

$$\mathcal{Q} = \left\{(x,z) \in \mathbb{R}^{n+q} \;\middle|\; \begin{pmatrix} x \\ z \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & \bar{Q} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} h_x \\ h_z \end{pmatrix}^\top \begin{pmatrix} x \\ z \end{pmatrix} + g \leq 0 \right\}.$$

Then for any $\bar{z} \in \{0,1\}^q$ we have

$$\mathcal{Q}|_{z=\bar{z}} = \{x \in \mathbb{R}^n \mid x^\top Q x + (2R\bar{z} + h_x)^\top x + g + h_z^\top \bar{z} + \bar{z}^\top \bar{Q} \bar{z} \leq 0\}.$$

Now since $\mathcal{Q}$ is a convex quadratic region, the matrix $Q$ must be positive semidefinite. Moreover, the matrix $Q$ is clearly independent of the choice of $\bar{z} \in \{0,1\}^q$.

It remains to show that the vector $2R\bar{z} + h_x$ decomposes into $Qw + v$ where $v \in \ker(Q)$ is independent of $\bar{z}$. We claim that $2R\bar{z} \in \operatorname{range}(Q)$. Decompose $2R\bar{z} = Qw + v$ for a unique vector $v \in \ker(Q)$. If $v \neq 0$, then for $\lambda < -\frac{\bar{z}^\top \bar{Q} \bar{z}}{v^\top v}$ we have

$$\begin{pmatrix} \lambda v \\ \bar{z} \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & \bar{Q} \end{pmatrix} \begin{pmatrix} \lambda v \\ \bar{z} \end{pmatrix} = \lambda^2 v^\top Q v + 2\bar{z}^\top R^\top (\lambda v) + \bar{z}^\top \bar{Q} \bar{z} = \lambda v^\top v + \bar{z}^\top \bar{Q} \bar{z} < 0,$$

a contradiction. Since $2R\bar{z} \in \operatorname{range}(Q)$, the vector $v$ depends only on $h_x$ and is thus independent of $\bar{z} \in \{0,1\}^q$. □

The next lemma can be seen as a converse of Lemma 1. We denote by $e_i \in \mathbb{R}^k$ the $i$th standard basis vector of $\mathbb{R}^k$.

**Lemma 2.** *Let $\mathcal{Q}_1, \ldots, \mathcal{Q}_k \subseteq \mathbb{R}^n$ be convex quadratic regions with the same shape. Then there exists a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+k}$ such that $\mathcal{Q}|_{z=e_i} = \mathcal{Q}_i$ for each $1 \leq i \leq k$.*

*Proof.* Suppose that each $\mathcal{Q}_i$ is described by

$$\mathcal{Q}_i = \{x \in \mathbb{R}^n \mid x^\top Q x + (2Qw^i + v)^\top x + g_i \leq 0\},$$

where $Q \succeq 0$ and $v \in \ker(Q)$. Set $\gamma_i \geq k (w^i)^\top Q w^i$ and $h_i = g_i - \gamma_i$, and define $R := \left(Qw^1 \mid \;\cdots\; \mid Qw^k\right)$, $\Lambda := \operatorname{diag}(\gamma_1, \ldots, \gamma_k)$, and $h^\top := (v^\top, h_1, \ldots, h_k)$. We claim that

$$\mathcal{Q} := \left\{(x,z) \in \mathbb{R}^{n+k} \;\middle|\; \begin{pmatrix} x \\ z \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & \Lambda \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + h^\top \begin{pmatrix} x \\ z \end{pmatrix} \leq 0 \right\}$$

is a convex quadratic region with the desired restriction property. Then

$$\mathcal{Q}|_{z=e_i} = \{x \in \mathbb{R}^n \mid x^\top Qx + (2Re_i + v)^\top x + h_i + \gamma_i \leq 0\}$$

and by construction $2Re_i = 2Qw^i$ and $h_i + \gamma_i = g_i$. Thus, $\mathcal{Q}|_{z=e_i} = \mathcal{Q}_i$.

We now show that $\mathcal{Q}$ is a convex quadratic region by demonstrating that matrix defining $\mathcal{Q}$ is positive semidefinite. Let $(x, z) \in \mathbb{R}^{n+k}$. We have that

$$\begin{aligned}
\begin{pmatrix} x \\ z \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & \Lambda \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} &= x^\top Qx + 2z^\top R^\top x + \sum_{i=1}^k \gamma_i z_i^2 \\
&= x^\top Qx + \sum_{i=1}^k \left(2(z_i Qw^i)^\top x + \gamma_i z_i^2\right) \\
&= \frac{1}{k} \sum_{i=1}^k \left(x^\top Qx + 2(Qkz_i w^i)^\top x + k\gamma_i z_i^2\right).
\end{aligned}$$

We show by choice of $\gamma_i$ that each summand $x^\top Qx + 2(Qkz_iw^i)^\top x + k\gamma_i z_i^2$ is nonnegative by completing the square. Note

$$x^\top Qx + 2(Qkz_iw^i)^\top x + k\gamma_i z_i^2 = (x + kz_iw^i)^\top Q(x + kz_iw^i) + (k\gamma_i - k^2(w^i)^\top Qw^i)z_i^2.$$

Now since $\gamma_i \geq k(w^i)^\top Qw^i$ we have expressed each summand as the sum of two non-negative numbers. In particular, each $x^\top Qx + 2(Qkz_iw^i)^\top x + k\gamma_i z_i^2 \geq 0$ and $\mathcal{Q}$ is a convex quadratic region. □

We note that Lemma 2 shows that a union of convex quadratic regions with the same shape have a binary lift to a convex quadratic region provided we intersect it with an appropriate polyhedron.

The proof of Theorem 2 is now a simple combination of the preceding lemmas. We note that for the construction of the extended formulation, we cannot use a system similar to that which appeared in Theorem 1 as it requires additional continuous variables.

*Proof of Theorem 2.* We start with sufficiency of the condition. Assume we have convex quadratic regions $\mathcal{Q}_1, \ldots, \mathcal{Q}_k \subseteq \mathbb{R}^n$ with the same shape, and polyhedra $\mathcal{P}_1, \ldots, \mathcal{P}_k \subseteq \mathbb{R}^n$ with the same recession cone and let $S$ be defined by (3). Then by Lemma 2, we obtain a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+k}$ such that $\mathcal{Q}|_{z=e_i} = \mathcal{Q}_i$ for each $1 \leq i \leq k$. We use a standard technique to obtain a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+k}$ such that $\mathcal{P}|_{z=e_i} = \mathcal{P}_i$ for $1 \leq i \leq k$ and $\mathcal{P}|_{z=\bar{z}} = \emptyset$ for $\bar{z} \in \{0,1\}^k - \{e_1, \ldots, e_k\}$. This technique is known as a Big-M formulation, and the existence of such a polyhedron is proved in Proposition 6.1 in [14]. It follows that $S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^n \times \{0,1\}^k))$.

It remains to show necessity. Let $\mathcal{Q} \subseteq \mathbb{R}^{n+q}$ be a convex quadratic region and $\mathcal{P} \subseteq \mathbb{R}^{n+q}$ be a polyhedron. Let $S := \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^n \times \{0,1\}^q))$. Then

$$S = \bigcup_{\bar{z} \in \{0,1\}^q} (\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}}.$$

Choose $\bar{z} \in \{0,1\}^q$ and note that $(\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}} = \mathcal{Q}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}}$. Then by Lemma 1 all $\mathcal{Q}|_{z=\bar{z}}$ are convex quadratic regions with the same shape. Since each polyhedron $\mathcal{P}|_{z=\bar{z}}$ has recession cone independent of $\bar{z}$ the theorem follows.

# 4 The Continuous Case

In this section we find an algebraic characterization of continuous convex quadratic representable sets, i.e., where $q = 0$ is fixed in (1). In the first part of this section we consider a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ and a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p}$. We proceed by computing explicitly the projection $\mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P})$ and in doing so come to the definitions for sets of Type 1 and Type 2. In particular, each set of Type 1 or Type 2 can be written as a finite union of $\mathcal{Q}_i \cap \mathcal{P}_i$ for convex quadratic regions $\mathcal{Q}_i$ and polyhedra $\mathcal{P}_i$. These definitions will be sufficient conditions for a set to be continuous convex quadratic representable. In order to show this, we demonstrate that every set of Type 1 or Type 2 has a lift to $\mathcal{Q} \cap \mathcal{P} \subseteq \mathbb{R}^{n+p}$ for some convex quadratic region $\mathcal{Q}$ and polyhedron $\mathcal{P}$.

Assume now that we are given a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$, a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p}$, and wish to compute $S := \mathrm{proj}_n(\mathcal{Q} \cap \mathcal{P})$. We begin by applying an invertible affine transformation to $\mathbb{R}^{n+p}$ that brings $\mathcal{Q}$ to a normalized form.

**Lemma 3.** *Let $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ be a convex quadratic region defined by*

$$\mathcal{Q} = \left\{ (x,y) \in \mathbb{R}^{n+p} \;\middle|\; \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & S \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_x \\ h_y \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g \leq 0 \right\}.$$

*Then there exists an invertible affine transformation $T : \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ that takes $\mathcal{Q}$ to a convex quadratic region $\mathcal{Q}' \subseteq \mathbb{R}^{n+p}$ such that* $\mathrm{proj}_n(\mathcal{Q}) = \mathrm{proj}_n(\mathcal{Q}')$ *and*

$$\mathcal{Q}' = \left\{ (x,y,t) \in \mathbb{R}^{n+k+(p-k)} \;\middle|\; \begin{pmatrix} x \\ y \\ t \end{pmatrix}^\top \begin{pmatrix} Q' & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} + \begin{pmatrix} h' \\ 0 \\ l \end{pmatrix}^\top \begin{pmatrix} x \\ y \\ t \end{pmatrix} + g' \leq 0 \right\}, \tag{4}$$

*where $k = \mathrm{rank}(S)$, $I_k$ is the $k \times k$ identity matrix, and either $l = e_1$ or $l = 0$.*

*Proof.* We will define $T$ as the composition of three invertible affine transformations. Since $S \succeq 0$, there exists an orthogonal $p \times p$ matrix $U$ such that $S = U^\top \Lambda U$ where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ and $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$. Suppose the first $k$ eigenvalues of $S$ are positive, and define $V = \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_k}, 1, \ldots, 1)$. Then $S = U^\top VEVU$ where $E$ is a diagonal matrix whose first $k$ diagonal entries are 1 and the remaining $p-k$ entries are 0.

Define the transformation $T' : \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ by multiplication on the left by the matrix

$$A = \begin{pmatrix} I & 0 \\ 0 & VU \end{pmatrix}.$$

Consider the change of coordinates defined by $(x,u)^\top = A(x,y)^\top$. Then $T'(\mathcal{Q})$ is described by

$$\left\{(x,u) \in \mathbb{R}^{n+p} \;\middle|\; \begin{pmatrix} x \\ u \end{pmatrix}^\top \begin{pmatrix} Q & RU^\top V^{-1} \\ V^{-1}UR^\top & E \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} h_x \\ V^{-1}Uh_y \end{pmatrix}^\top \begin{pmatrix} x \\ u \end{pmatrix} + g \leq 0 \right\}.$$

Note now that the matrix defining the quadratic region $T'(\mathcal{Q})$ is positive semidefinite. This implies that any diagonal entry being 0 forces the entire corresponding row and column to be 0 as well. Let $B$ denote the first $k$ columns of $RU^\top V^{-1}$. Then

$$\begin{pmatrix} Q & RU^\top V^{-1} \\ V^{-1}UR^\top & E \end{pmatrix} = \begin{pmatrix} Q & B & 0 \\ B^\top & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define $T'' : \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ by multiplication on the left by the invertible matrix

$$C = \begin{pmatrix} I_n & 0 & 0 \\ B^\top & I_k & 0 \\ 0 & 0 & I_{p-k} \end{pmatrix}.$$

Consider the change of coordinates defined by $(x,v,w)^\top = C(x,u)^\top$. Then $T''(T'(\mathcal{Q}))$ is described by

$$\left\{(x,v,w) \in \mathbb{R}^{n+k+(p-k)} \;\middle|\; \begin{pmatrix} x \\ v \\ w \end{pmatrix}^\top \begin{pmatrix} Q' & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \\ w \end{pmatrix} + \begin{pmatrix} h' \\ h_v \\ h_w \end{pmatrix}^\top \begin{pmatrix} x \\ v \\ w \end{pmatrix} + g \leq 0 \right\},$$

where $Q' := Q - BB^\top$, $h' := h_x - B(V^{-1}Uh_y)_+$, and $\begin{pmatrix} h_v \\ h_w \end{pmatrix} := V^{-1}Uh_y$.

Finally, define the affine transformation $L : \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ by $L(x,v,w) = H(x,v,w)^\top + r$ where

$$H = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & M \end{pmatrix},$$

and $M$ is either an invertible matrix such that $(M^{-1})^\top h_w = e_1$ if $h_w \neq 0$ or $M = I_{p-k}$ if $h_w = 0$, and $r = (0, -\frac{1}{2}h_v, 0)^\top$. We now change coordinates $(x,y,t)^\top = H(x,v,w)^\top + r$.

Define $T = L \circ T'' \circ T'$. Then $T$ is an invertible affine transformation that takes $\mathcal{Q}$ to $\mathcal{Q}' := T(\mathcal{Q})$ described by (4). Note that $T$ is determined by multiplication by a matrix whose first $n$ rows are $\begin{pmatrix} I_n | & 0 \end{pmatrix}$ and a vector $r$ whose first $n$ entries are zero. This implies that $\text{proj}_n(\mathcal{Q}) = \text{proj}_n(\mathcal{Q}')$ and the proof is complete. □

Note that by Lemma 3, without loss of generality, we may assume that $\mathcal{Q}$ is described by (4). We can further simplify the structure of $\mathcal{Q}$ by projecting out all variables $t_i$ that do not explicitly appear in the description of $\mathcal{Q}$.

**Lemma 4.** *Assume that $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ is a convex quadratic region described by* (4) *and that $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ is a polyhedron. If $l = 0$ then $\operatorname{proj}_{n+k}(\mathcal{Q} \cap \mathcal{P}) = \mathcal{Q}' \cap \mathcal{P}'$ where $\mathcal{P}' = \operatorname{proj}_{n+k}(\mathcal{P})$ and $\mathcal{Q}'$ is described by*

$$\left\{(x, y) \in \mathbb{R}^{n+k} \;\middle|\; \begin{pmatrix} x \\ y \end{pmatrix}^{\top} \begin{pmatrix} Q & 0 \\ 0 & I_k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix}^{\top} \begin{pmatrix} x \\ y \end{pmatrix} + g \leq 0\right\}. \tag{5}$$

*If $l = e_1$, then $\operatorname{proj}_{n+k+1}(\mathcal{Q} \cap \mathcal{P}) = \mathcal{Q}' \cap \mathcal{P}'$ where $\mathcal{P}' = \operatorname{proj}_{n+k+1}(\mathcal{P})$ and $\mathcal{Q}'$ is described by*

$$\left\{(x, y, t) \in \mathbb{R}^{n+k+1} \;\middle|\; \begin{pmatrix} x \\ y \\ t \end{pmatrix}^{\top} \begin{pmatrix} Q & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} + \begin{pmatrix} h \\ 0 \\ 1 \end{pmatrix}^{\top} \begin{pmatrix} x \\ y \\ t \end{pmatrix} + g \leq 0\right\}. \tag{6}$$

*Proof.* Let $r = k$ in the case of (5) and $r = k + 1$ in the case of (6). It suffices to show that $\operatorname{proj}_{n+r}(\mathcal{Q} \cap \mathcal{P}) = \operatorname{proj}_{n+r}(\mathcal{Q}) \cap \operatorname{proj}_{n+r}(\mathcal{P})$. Then for any $\bar{x} \in \operatorname{proj}_{n+r}(\mathcal{Q}) \cap \operatorname{proj}_{n+r}(\mathcal{P})$ there exists $s^1, s^2 \in \mathbb{R}^{p-r}$ such that $(\bar{x}, s^1) \in \mathcal{Q}$ and $(\bar{x}, s^2) \in \mathcal{P}$. Since $e_{n+r+j} \in \operatorname{lin}(\mathcal{Q})$ for each $j \geq 1$ we have $(\bar{x}, s^2) \in \mathcal{Q}$ and hence $\bar{x} \in \operatorname{proj}_{n+r}(\mathcal{Q} \cap \mathcal{P})$. The reverse containment is clear. □

Then by Lemma 4, without loss of generality, we may assume that $\mathcal{Q}$ is described by either (5) or (6).

We now construct a family of affine spaces that will simplify the computation of $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P})$. These affine spaces will form a sort of skeleton of the region $\mathcal{Q} \cap \mathcal{P}$ that will contain all the essential information of $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P})$. We will make use of the following observations whose short proofs we include for completion.

**Observation 1.** *Let $q(x) = x^\top Q x + h^\top x + g$ be a quadratic polynomial in $n$ variables where $Q$ is a positive semidefinite matrix. Then $q(x)$ has a minimum on $\mathbb{R}^n$ if and only if $h \in \operatorname{range}(Q)$. In this case, the set of minimizers of $q(x)$ is $\{x \in \mathbb{R}^n \mid 2Qx + h = 0\}$.*

*Proof.* Assume $h \notin \operatorname{range}(Q)$. Then since $Q$ is symmetric, we can write $h = Qw + v$ with $Qv = 0$ and $v \neq 0$. Consider $x(t) = -tv$ for $t \in \mathbb{R}$. Then we have

$$q(x(t)) = h^\top x(t) + g = -tv^\top v + g.$$

Since $v \neq 0$, we see that $q(x(t)) \to -\infty$ as $t \to +\infty$. Thus, $q(x)$ has no minimum on $\mathbb{R}^n$.

We now prove the reverse direction. Since $Q$ is positive semidefinite, the function $q(x)$ attains its minimum at $\bar{x}$ if and only if $\bar{x}$ solves $\nabla q(x) = 2Qx + h = 0$. This set is nonempty since $h \in \operatorname{range}(Q)$. □

**Observation 2.** *Let $\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Q x + h^\top x + g \leq 0\}$ be a non-empty convex quadratic region. Then*

$$\operatorname{rec}(\mathcal{Q}) = \{r \in \mathbb{R}^n \mid Qr = 0,\ h^\top r \leq 0\}.$$

*Proof.* Let $r \in \mathbb{R}^n$ such that $Qr = 0$ and $h^\top r \leq 0$. Fix $\bar{x} \in \mathcal{Q}$ and $\lambda \geq 0$. Then $(\bar{x}+\lambda r)^\top Q(\bar{x}+\lambda r) + h^\top(\bar{x}+\lambda r) + g = \bar{x}^\top Q\bar{x} + h^\top\bar{x} + g + \lambda h^\top r \leq 0$ and $\bar{x}+\lambda r \in \mathcal{Q}$. It follows that $r \in \operatorname{rec}(\mathcal{Q})$.

Assume now that $r \in \mathbb{R}^n$ either satisfies $Qr \neq 0$ or $Qr = 0$ and $h^\top r > 0$. Fix $\bar{x} \in \mathcal{Q}$. Then for any $\lambda \geq 0$ we have

$$(\bar{x}+\lambda r)^\top Q(\bar{x}+\lambda r)+h^\top(\bar{x}+\lambda r)+g = \lambda^2 r^\top Qr+\lambda(h+2Q\bar{x})^\top r+\bar{x}^\top Q\bar{x}+h^\top\bar{x}+g,$$

a polynomial in $\lambda$. Since $Q \succeq 0$, as $\lambda \to \infty$, this polynomial increases indefinitely. Thus, $r \notin \operatorname{rec}(\mathcal{Q})$. $\square$

**Proposition 1.** *Assume that $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ is a convex quadratic region described by* (5) *or* (6) *and that $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ is a polyhedron. Then either $\operatorname{proj}_n(\mathcal{Q}\cap\mathcal{P}) = \operatorname{proj}_n(\mathcal{P})$ or there exists a finite collection $\mathcal{L}$ of affine spaces such that*

$$\operatorname{proj}_n(\mathcal{Q}\cap\mathcal{P}) = \bigcup_{L\in\mathcal{L}} \operatorname{proj}_n(\mathcal{Q}\cap\mathcal{P}\cap L). \tag{7}$$

*Moreover, each $L \in \mathcal{L}$ has dimension $n$ and can be described by a system $Fx + Gy = d$ where $G$ is an invertible $p\times p$ matrix.*

*Proof.* Assume first that $\mathcal{Q}$ is described by (5) and let

$$q(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & 0 \\ 0 & I_k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g.$$

Suppose that $\mathcal{P} = \{(x,y) \in \mathbb{R}^{n+p} \mid (a^i)^\top x + (b^i)^\top y \leq c_i \text{ for all } i \in I\}$ where $I$ is some finite index set. Define $\mathcal{J}$ to be the collection of subsets $J \subseteq I$ with $|J| \leq p$ and such that the set $\{b^i\}_{i\in J}$ is linearly independent. For each $J \in \mathcal{J}$ we construct an affine space $L_J$ to include in $\mathcal{L}$. We construct $L_J$ so that for every fixed $\bar{x} \in \mathbb{R}^n$, the unique point $(\bar{x}, y^J_{\bar{x}}) \in L_J$ minimizes $q(\bar{x},y)$ over the affine space determined by setting to equality the inequalities indexed by $J$ and fixing $x = \bar{x}$.

We first note that $\emptyset \in \mathcal{J}$ trivially. In the case that $J = \emptyset$, we define $L_J = \{(x,y) \in \mathbb{R}^{n+p} \mid y = 0\}$. Then for any fixed $\bar{x} \in \mathbb{R}^n$ the point $y = 0$ minimizes $q(\bar{x},y)$ on $\mathbb{R}^p$. We will have need to refer to following construction again, so we emphasize it here.

**Construction of a Minimizing Affine Space.** Consider a nonempty set $J \in \mathcal{J}$, say $|J| = m$, and define $R, U$, and $l$ to be the matrices with rows $(a^i)^\top$, $(b^i)^\top$, and $c_i$ for $i \in J$, respectively. Consider the affine space $K_J$ defined by $Rx + Uy = l$.

Since $m \leq p$ we know $U$ has rank $m$ and we can partition the $y$ variables into new variables $u$ and $v$ where the $v$ variables correspond to columns of $U$ that define a full rank submatrix. This division into $(u,v) \in \mathbb{R}^{p-m}\times\mathbb{R}^m$ causes $K_J$ to be described by $Rx + Su + Tv = l$ where $T$ is invertible. Substitute $v = T^{-1}(l - Rx - Su)$ into the polynomial $q(x,u,v)$ and fix a point $\bar{x} \in \mathbb{R}^n$ to obtain the polynomial $q(\bar{x},u)$ defined by

$$u^\top(I + S^\top(T^{-1})^\top T^{-1}S)u + 2(S^\top(T^{-1})^\top T^{-1}R\bar{x} + 2S^\top(T^{-1})^\top T^{-1}l)^\top u + g(\bar{x}).$$

Now since $I+S^\top(T^{-1})^\top T^{-1}S$ is positive definite, by Observation 1, the unique minimum of $q(\bar{x},u)$ is the point satisfying

$$2(I+S^\top(T^{-1})^\top T^{-1}S)u+2(S^\top(T^{-1})^\top T^{-1}R\bar{x}-S^\top(T^{-1})^\top T^{-1}l)=0.$$

Note that this minimum depends linearly on $\bar{x}$. We thus define $L_J$ to be the affine space determined by

$$\begin{aligned} S^\top(T^{-1})^\top T^{-1}Rx+(I+S^\top(T^{-1})^\top T^{-1}S)u &= S^\top(T^{-1})^\top T^{-1}l \\ Rx+Su+Tv &= l. \end{aligned} \tag{8}$$

Since $I+S^\top(T^{-1})^\top T^{-1}S$ and $T$ are invertible matrices, $L_J$ is an affine space of dimension $n$ and is described by a system $Fx+Gy=d$ where $G$ is an invertible $p\times p$ matrix. This marks the end of the construction. †

Set

$$\mathcal{L}=\{L_J \mid J\in\mathcal{J}\}. \tag{9}$$

We claim that $\mathcal{L}$ satisfies (7). It suffices to show that for any point $\bar{x}\in\operatorname{proj}_n(\mathcal{Q}\cap\mathcal{P})$, there exists $\bar{y}\in\mathbb{R}^p$ and $L\in\mathcal{L}$ such that $(\bar{x},\bar{y})\in\mathcal{Q}\cap\mathcal{P}\cap L$.

Let $\bar{x}\in\operatorname{proj}_n(\mathcal{Q}\cap\mathcal{P})$. Then there exists $y^0\in\mathbb{R}^p$ such that $(\bar{x},y^0)\in\mathcal{Q}\cap\mathcal{P}$. This implies that $q(\bar{x},y^0)\leq 0$ and since $q(\bar{x},0)$ minimizes $q(\bar{x},y)$ on $\mathbb{R}^p$ we have $(\bar{x},0)\in\mathcal{Q}$ as well. If $y^0=0$, we may choose $L_J$ corresponding to $J=\emptyset$ and we are done. Otherwise, the line segment joining $(\bar{x},y^0)$ and $(\bar{x},0)$ is completely contained in $\mathcal{Q}$. Then by moving along this line segment from $(\bar{x},y^0)$ toward $(\bar{x},0)$ and inside $\mathcal{P}$ we either reach the point $(\bar{x},0)$ or stop at a point $(\bar{x},y^1)\in\mathcal{P}$. Then there exists an inequality $(a^i)^\top x+(b^i)^\top y\leq c_i$ with $b^i\neq 0$ that is satisfied at equality by $(\bar{x},y^1)$ and is not satisfied by $(\bar{x},0)$. We then set $J=\{i\}$ and continue this sliding process recursively.

Assume that we are at the point $(\bar{x},y^k)$ with current index set $J$. We now consider the line segment joining $(\bar{x},y^k)$ and $(\bar{x},y^J_{\bar{x}})$. Since $(\bar{x},y^J_{\bar{x}})$ is the minimizer of $q(\bar{x},y)$ on $K_J$, this line segment is contained in $\mathcal{Q}\cap K_J$. Again, slide the point $(\bar{x},y^k)$ toward $(\bar{x},y^J_{\bar{x}})$ inside $\mathcal{P}$ and we either reach the point $(\bar{x},y^J_{\bar{x}})$ or stop at a point $(\bar{x},y^{k+1})\in\mathcal{P}$. Then there exists an inequality $(a^j)^\top x+(b^j)^\top y\leq c_j$ with $b^j\notin\operatorname{Span}(\{b^i\}_{i\in J})$ that is satisfied at equality by $(\bar{x},y^{k+1})$ and is not satisfied by $(\bar{x},y^J_{\bar{x}})$. We update $J$ to include $j$ and repeat this process.

The end result is that we find a point $(\bar{x},\bar{y})\in\mathcal{Q}\cap\mathcal{P}\cap L_J$ for some $J\in\mathcal{J}$. In fact, either we hit a point $(\bar{x},y^J_{\bar{x}})$ at some iteration or after applying the procedure $p$ times we restrict ourselves to an $n$-dimensional affine space, which by construction must be in $\mathcal{L}$.

Now assume $\mathcal{Q}$ is described by (6). There is one degenerate case to consider. Note that for any fixed $\bar{x}\in\mathbb{R}^n$ we have $\operatorname{rec}(\mathcal{Q}|_{x=\bar{x}})=\{(0,-\lambda)\in\mathbb{R}^{p+1}\mid\lambda\geq 0\}$ by Observation 2. Moreover, for any $(\bar{x},\bar{y})\in\mathbb{R}^{n+p}$ there exists $\bar{t}\in\mathbb{R}$ such that $(\bar{x},\bar{y},\bar{t})\in\mathcal{Q}$. To see this, simply take $\bar{t}\leq-(\bar{x}^\top Q\bar{x}+\bar{y}^\top\bar{y}+h^\top\bar{x}+g)$. Suppose that $(0,-1)\in\operatorname{rec}(\mathcal{P}|_{x=\bar{x}})$ for every $\bar{x}\in\mathbb{R}^n$. We claim that $\operatorname{proj}_n(\mathcal{Q}\cap\mathcal{P})=\operatorname{proj}_n(\mathcal{P})$. Let $\bar{x}\in\operatorname{proj}_n(\mathcal{P})$ so that there exists $(\bar{y},\bar{t})\in\mathbb{R}^{p+1}$ such that $(\bar{x},\bar{y},\bar{t})\in\mathcal{P}$. Then, by the note above, for sufficiently large $\lambda\geq 0$ we have $(\bar{x},\bar{y},\bar{t}-\lambda)\in\mathcal{Q}\cap\mathcal{P}$ and $\bar{x}\in\operatorname{proj}_n(\mathcal{Q}\cap\mathcal{P})$.

The remaining case when $\mathcal{Q}$ is described by (6) and $(0,-1) \notin \operatorname{rec}(\mathcal{P}|_{x=\bar{x}})$ for any $\bar{x} \in \mathbb{R}^n$ follows similarly to the case where $\mathcal{Q}$ is described by (5). We make note of the necessary changes in the proof. Adjust $\mathcal{J}$ to be the set of subsets $J \subseteq I$ such that $|J| \leq p+1$ and where $K_J$ is described by a system $Rx + Uy + Vt = l$ where $(U \mid V)$ is of full rank and at least one entry of $V$ is negative. This guarantees that for each fixed $\bar{x} \in \mathbb{R}^n$ the polynomial $q(\bar{x}, y)$ has a minimum on $K_J|_{x=\bar{x}}$, the same condition that we needed before. Note that in this case the empty set is not a member of $\mathcal{J}$.

We demonstrate how to modify the first step of the recursive descent described in the previous case. After this initial step the recursion continues exactly as in the general step detailed above. Let $\bar{x} \in \operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P})$. Then there exists $(y^0, t^0) \in \mathbb{R}^p$ such that $(\bar{x}, y^0, t^0) \in \mathcal{Q} \cap \mathcal{P}$. Since $(0,-1) \notin \operatorname{rec}(\mathcal{P}|_{x=\bar{x}})$, the ray based at $(\bar{x}, y^0, t^0)$ and directed along $(0,0,-1)$ cannot be completely contained in $\mathcal{P}$. In particular, moving in the direction $(0,0,-1)$ from the point $(\bar{x}, y^0, t^0)$ and inside $\mathcal{P}$ we stop at a point $(\bar{x}, y^1, t^1) \in \mathcal{P}$. Then there exists an inequality $(a^i)^\top x + (b^i)^\top y + v_i t \leq c_i$ with $v_i < 0$ that is satisfied at equality by $(\bar{x}, y^1, t^1)$. We now set $J = \{i\}$ and the recursion process continues identically as before. $\square$

The family $\mathcal{L}$ of affine spaces defined in Proposition 1 allows us to explicitly compute the set $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P})$. We do so by considering each set $\mathcal{Q} \cap \mathcal{P} \cap L$ in turn. In the next lemma we will see that for each $L \in \mathcal{L}$, the projection $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L) = \mathcal{Q}_L \cap \mathcal{P}_L$ for some convex quadratic region $\mathcal{Q}_L \subseteq \mathbb{R}^n$ and polyhedron $\mathcal{P}_L \subseteq \mathbb{R}^n$. This implies that the set $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P})$ is a finite union of the sets $\mathcal{Q}_L \cap \mathcal{P}_L$. However, in contrast to Theorem 1 in [5], the $\mathcal{Q}_L$ and $\mathcal{P}_L$ appearing in the projection cannot be arbitrary. We will see that they share a common structure. An understanding of this structure is essential to finding an extended formulation and thus obtaining a full algebraic characterization. This compatibility requirement is captured in our definition of sets of Type 1 and Type 2.

From here on, we compute an algebraic description of $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P})$ where $\mathcal{Q}$ is described by (5) or (6). The region resulting from $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P})$ in the case (5) will be called a set of *Type 1* and in the case of (6) a set of *Type 2*.

We are thus interested in computing $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L)$ where $L \in \mathcal{L}$. We claim that $\operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L) = \operatorname{proj}_n(\mathcal{Q} \cap L) \cap \operatorname{proj}_n(\mathcal{P} \cap L)$. Let $\bar{x} \in \operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P}) \cap \operatorname{proj}_n(\mathcal{P} \cap L)$. Then there exists $y^1, y^2 \in \mathbb{R}^p$ such that $(\bar{x}, y^1) \in \mathcal{Q} \cap L$ and $(\bar{x}, y^2) \in \mathcal{P} \cap L$. Now since $L$ is defined by $Fx + Gy = d$ with $G$ an invertible $p \times p$ matrix, we have $y^i = G^{-1}(d - F\bar{x})$ for $i = 1, 2$. In particular, $y^1 = y^2$ and we have that $\bar{x} \in \operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L)$. The reverse containment is obvious.

Therefore, a description of the sets $\operatorname{proj}_n(\mathcal{Q} \cap L)$ and $\operatorname{proj}_n(\mathcal{P} \cap L)$ is of particular interest to us as they serve as the base regions making up continuous convex quadratic representable sets. We define two functions $f_1, f_2$ that take as input a convex quadratic region $\mathcal{Q}$ of the form (5) or (6), respectively, and a special $n$-dimensional affine space $L$ and output a convex quadratic region, which we will show to be the projection onto $\mathbb{R}^n$ of the set $\mathcal{Q} \cap L$.

Let $\mathcal{Q}$ be a convex quadratic region described by (5) and $L$ an affine space described by $Fx+Gy=d$ with $G$ an invertible $p\times p$ matrix. We define $f_1[\mathcal{Q},L]$ to be the set of $x\in\mathbb{R}^n$ satisfying

$$x^\top(Q+F^\top(G^{-1})^\top G^{-1}F)x+(h-2F^\top(G^{-1})^\top G^{-1}d)^\top x+d^\top(G^{-1})^\top G^{-1}d+g\le 0.$$

Let $\mathcal{Q}$ be a convex quadratic region described by (6) and $L$ an affine space described by $Fx+G(y,t)^\top=d$ where $G$ is an invertible $(p+1)\times(p+1)$ matrix. We define $f_2[\mathcal{Q},L]$ to be the set of $x\in\mathbb{R}^n$ satisfying

$$\begin{aligned}x^\top(Q+F^\top(G^{-1})^\top E_kG^{-1}F)x+(h-F^\top(G^{-1})^\top e_{k+1}-2F^\top(G^{-1})^\top E_kG^{-1}d)^\top x\\ +d^\top(G^{-1})^\top E_kG^{-1}d+((G^{-1})^\top e_{k+1})^\top d+g\le 0,\end{aligned}$$

where $E_k$ is the $(k+1)\times(k+1)$ matrix with principal $k\times k$ minor the identity matrix and zero elsewhere. We note that $f_1[\mathcal{Q},L]$ and $f_2[\mathcal{Q},L]$ are convex quadratic regions in $\mathbb{R}^n$, since the matrices defining them are each the sum of two positive semidefinite matrices.

Assume now that $\mathcal{Q}$ is described by (5). We show that $\operatorname{proj}_n(\mathcal{Q}\cap L)=f_1[\mathcal{Q},L]$. Define the invertible linear transformation $T_A:\mathbb{R}^{n+p}\to\mathbb{R}^{n+p}$ by multiplication on the left by the matrix

$$A=\begin{pmatrix}I_n & 0\\ F & G\end{pmatrix}.$$

Then, we have that $\operatorname{proj}_n(\mathcal{Q}\cap L)=T_A(\mathcal{Q})|_{y=d}=f_1[\mathcal{Q},L]$. A similar proof shows that $\operatorname{proj}_n(\mathcal{Q}\cap L)=f_2[\mathcal{Q},L]$ when $\mathcal{Q}$ is described by (6).

Similarly, we define a function $\Pi$ that takes as input a polyhedron $\mathcal{P}$ in $\mathbb{R}^{n+r}$ and a special $n$-dimensional affine space and outputs a polyhedron in $\mathbb{R}^n$. Let $\mathcal{P}=\{x\in\mathbb{R}^{n+r}\mid (a^i)^\top x+(b^i)^\top y\le c_i \text{ for all } i\in I\}$ be a polyhedron and $L$ an affine space described by $Fx+Gy=d$ with $G$ an invertible $r\times r$ matrix. We define $\Pi[\mathcal{P},L]$ to be the polyhedron

$$\{x\in\mathbb{R}^n\mid (a^i)^\top x+(b^i)^\top G^{-1}(d-Fx)\le c_i \text{ for all } i\in I\}.$$

It is immediate from the substitution $y=G^{-1}(d-Fx)$ that $\operatorname{proj}_n(\mathcal{P}\cap L)=\Pi[\mathcal{P},L]$. Thus, we have established the following lemma.

**Lemma 5.** *Let $\mathcal{Q}\subseteq\mathbb{R}^{n+p}$ be a convex quadratic region given by* (5) *or* (6) *and $\mathcal{P}\subseteq\mathbb{R}^{n+p}$ a polyhedron. Let $\mathcal{L}$ be the family defined in Proposition 1. For each $L\in\mathcal{L}$ define $\mathcal{P}_L=\Pi[\mathcal{P},L]$ and either $\mathcal{Q}_L=f_1[\mathcal{Q},L]$ in the case of* (5)*, or $\mathcal{Q}_L=f_2[\mathcal{Q},L]$ in the case of* (6)*. Then*

$$\operatorname{proj}_n(\mathcal{Q}\cap\mathcal{P})=\bigcup_{L\in\mathcal{L}}\mathcal{Q}_L\cap\mathcal{P}_L.$$

We can use the algebraic description from Lemma 5 to complete our characterization. We are now ready for our technical definitions of Type 1 and Type 2.

Let $S\subseteq\mathbb{R}^n$. We say that $S$ is a set of *Type 1* if there exists a convex quadratic region $\bar{\mathcal{Q}}=\{x\in\mathbb{R}^n\mid x^\top Qx+h^\top x+g\le 0\}$, an integer $k\ge 0$, a finite index set $I$, and vectors $(a^i, b^i, c_i) \in \mathbb{R}^{n+k+1}$ for each $i \in I$ with the following compatibility structure.

Let $\mathcal{J}$ be the collection of subsets $J \subseteq I$ with $|J| \leq k$ such that the set $\{b^i\}_{i \in J}$ is linearly independent. Then for each nonempty $J \in \mathcal{J}$ we define the affine space $L_J \subseteq \mathbb{R}^{n+k}$ to be the output of the construction of a minimizing affine space found in the proof of Proposition 1. These objects are required to satisfy

$$S = (\bar{\mathcal{Q}} \cap_{i \in I} \{x \in \mathbb{R}^n \mid (a^i)^\top x \leq c^i\}) \bigcup_{J \in \mathcal{J}} (\mathcal{Q}_{L_J} \cap \mathcal{P}_{L_J})$$

where each $\mathcal{P}_{L_J} = \Pi[\mathcal{P}, L_J]$ and each $\mathcal{Q}_{L_J} = f_1[\mathcal{Q}, L_J]$.

The definition of a set of *Type 2* is exactly as above, except that $S$ is required to satisfy

$$S = \bigcup_{J \in \mathcal{J}} (\mathcal{Q}_{L_J} \cap \mathcal{P}_{L_J})$$

where each $\mathcal{P}_{L_J} = \Pi[\mathcal{P}, L_J]$ and each $\mathcal{Q}_{L_J} = f_2[Q, L_J]$.

**Theorem 3.** *Let $S \subseteq \mathbb{R}^n$. Then $S$ is continuous convex quadratic representable if and only if $S$ is a set of Type 1 or Type 2.*

*Proof.* Assume first that $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ is a convex quadratic region and $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ is a polyhedron. Then Lemma 3, Lemma 4, Proposition 1, and Lemma 5 show that $S$ is of Type 1 or Type 2.

Assume now that $S$ is a set of Type 1. Consider the convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+k}$ described by (5), i.e.,

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & 0 \\ 0 & I_k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g \leq 0,$$

and the polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+k}$ described as

$$\{(x, y) \in \mathbb{R}^{n+k} \mid (a^i)^\top x + (u^i)^\top y \leq c^i \text{ for all } i \in I\}.$$

Then Proposition 1 and Lemma 5 show that $S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$.

The case of $S$ being a set of Type 2 is identical to the case of Type 1, save for the construction of $\mathcal{Q}$ satisfying (6) instead of (5). ☐

# 5 The Mixed Binary Case

In this section, we combine the results of Sections 3 and 4 to state a characterization theorem for sets $S \subseteq \mathbb{R}^n$ that are mixed binary convex quadratic representable.

Let $S \subseteq \mathbb{R}^n$ be a set of Type 1 (or 2). Then $S$ is determined by the data of a convex quadratic region $\bar{\mathcal{Q}} \subseteq \mathbb{R}^n$, an integer $k \geq 0$, an index set $I$, and vectors $(a^i, b^i, c^i) \in \mathbb{R}^{n+k+1}$ for $i \in I$.

Given two sets $S, S' \subseteq \mathbb{R}^n$ both of Type 1 (resp. both of Type 2), we say that $S$ and $S'$ have the *same structure* if the data determining $S$ and $S'$ as sets of Type 1 (resp. Type 2) can be chosen so that

(i) $k = k'$,

(ii) $\bar{\mathcal{Q}}$ and $\bar{\mathcal{Q}}'$ have the same shape,

(iii) $I = I'$ and $(a^i, b^i) = (a'^i, b'^i)$ for each $i \in I$.

We can now state and prove our characterization theorem.

**Theorem 4.** *Let $S \subseteq \mathbb{R}^n$. Then $S$ is mixed binary convex quadratic representable if and only if there exist sets $S_1, \ldots, S_r \subseteq \mathbb{R}^n$ all of Type 1 (or all of Type 2) with the same structure, such that $S = \cup_{i=1}^r S_i$.*

*Proof.* Assume first that there exist sets $S_1, \ldots, S_r \subseteq \mathbb{R}^n$ of Type 1 all with the same structure such that $S = \cup_{i=1}^r S_i$. Then by Theorem 3 there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^{n+k}$ and polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^{n+k}$ for $i = 1, \ldots, r$ such that $S_i = \text{proj}_n(\mathcal{Q}_i \cap \mathcal{P}_i)$. Moreover, it follows from the construction given in the proof of Theorem 3 that all the $\mathcal{Q}_i$ have the same shape and all $\mathcal{P}_i$ have the same recession cone. It follows by applying Theorem 2 to $\cup_{i=1}^r (\mathcal{Q}_i \cap \mathcal{P}_i)$ that there exists a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+k+r}$ and a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+k+r}$ such that

$$\bigcup_{i=1}^r (\mathcal{Q}_i \cap \mathcal{P}_i) = \text{proj}_{n+k}(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+k} \times \{0,1\}^r)).$$

Now,

$$\text{proj}_n(\bigcup_{i=1}^r (\mathcal{Q}_i \cap \mathcal{P}_i)) = \bigcup_{i=1}^r \text{proj}_n(\mathcal{Q}_i \cap \mathcal{P}_i) = \bigcup_{i=1}^r S_i = S.$$

In particular, $S$ is mixed binary convex quadratic representable. The proof for sets of Type 2 follows similarly.

For the reverse direction, let $\mathcal{Q} \subseteq \mathbb{R}^{n+p+q}$ be a convex quadratic region and $\mathcal{P} \subseteq \mathbb{R}^{n+p+q}$ be a polyhedron and set

$$S := \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0,1\}^q)).$$

Then by allowing $\bar{z}$ to vary over $\{0,1\}^q$ and Lemma 1 we have

$$S = \bigcup_{\bar{z} \in \{0,1\}^q} \text{proj}_n(\mathcal{Q}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}}),$$

where each $\mathcal{Q}|_{z=\bar{z}}$ has the same shape and each $\mathcal{P}|_{z=\bar{z}}$ has the same recession cone. We use the following technical claim to complete the proof of the theorem.

**Claim 1.** *Let $\mathcal{Q}, \mathcal{Q}' \subseteq \mathbb{R}^{n+p}$ be two convex quadratic regions with the same shape and $\mathcal{P}, \mathcal{P}' \subseteq \mathbb{R}^{n+p}$ be polyhedra with the same recession cone. Then $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ and $\text{proj}_n(\mathcal{Q}' \cap \mathcal{P}')$ are both sets of Type 1 (or both sets of Type 2) with the same structure.*

*Proof of claim.* Let $S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ and $S' = \text{proj}_n(\mathcal{Q}' \cap \mathcal{P}')$. We first normalize $\mathcal{Q}$ and $\mathcal{Q}'$ as in Lemma 3. Note that an invertible affine transformation takes two convex quadratic regions with the same shape to two convex quadratic regions with the same shape. Similarly, an invertible transformation preserves equality of the recession cones of two polyhedron.

Thus, we can assume that $\mathcal{Q}$ and $\mathcal{Q}'$ have the same shape and are described by (4). We can now apply Lemma 4 and further assume that $\mathcal{Q}$ and $\mathcal{Q}'$ are both described by (5), or both by (6), and still having the same shape. Moreover, since $\text{proj}_{n+k}(\text{rec}(\mathcal{P})) = \text{rec}(\text{proj}_{n+k}(\mathcal{P}))$ we may still assume that $\mathcal{P}$ and $\mathcal{P}'$ have the same recession cone.

Assume now that $\mathcal{Q}$ and $\mathcal{Q}'$ are described by (5) having the same shape and that $\mathcal{P}$ and $\mathcal{P}'$ have the same recession cone. It is well known that there exists a matrix $M \in \mathbb{R}^{m\times(n+k)}$ and vectors $r, r' \in \mathbb{R}^m$ such that $\mathcal{P} = \{z \in \mathbb{R}^{n+k} \mid Mz \leq r\}$ and $\mathcal{P}' = \{z \in \mathbb{R}^{n+k} \mid Mz \leq r'\}$. It now follows that $S$ and $S'$ are both sets of Type 1 (or both of Type 2) with the same structure. $\diamond$

By Claim 1, it follows that the sets $\text{proj}_n(\mathcal{Q}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}})$ are all sets of Type 1 (or all sets of Type 2) with the same structure. $\square$

# 6 Toward a Geometric Characterization

The algebraic characterizations in Section 4 of continuous convex quadratic representable sets and in Section 5 of mixed binary convex quadratic representable sets lead to a natural question. Are there geometric conditions that characterize continuous and mixed binary convex quadratic representable sets? In this section, we focus on what these algebraic characterizations imply concerning a geometric description of representable sets.

Consider a continuous convex quadratic representable set $S \subseteq \mathbb{R}^n$. As a consequence of Theorem 3, there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$ and polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$ for $i = 1, \ldots, k$ such that $S = \cup_{i=1}^k \mathcal{Q}_i \cap \mathcal{P}_i$. Since $S$ is representable it can be realized as the projection of a convex set which implies that $S$ must be convex as well.

It is unclear whether these two obvious necessary conditions are in fact sufficient as well. This leads us to the following question.

**Question 1.** *Let $S \subseteq \mathbb{R}^n$. Is it true that $S$ is continuous convex quadratic representable if and only if $S$ is convex and there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$ and polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$ for $i = 1, \ldots, k$ such that*

$$S = \bigcup_{i=1}^{k} \mathcal{Q}_i \cap \mathcal{P}_i \quad ? \tag{10}$$

The main difficulty in establishing a positive answer to this question is finding an extended formulation for a set $S$ given by (10). As a step in this direction, given a finite collection of convex quadratic regions $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$ in $\mathbb{R}^n$ we can show that there exist a convex quadratic region $\mathcal{Q}$ in $\mathbb{R}^{n+k(n+1)}$ and affine spaces

$L_1, \dots, L_k$ in $\mathbb{R}^{n+k(n+1)}$, described by $F_i x + G_i y = d^i$ with each $G_i$ an invertible matrix, such that $\mathcal{Q}_i = \text{proj}_n(\mathcal{Q} \cap L_i)$ for $i = 1, \dots, k$. It is unclear whether this construction allows for a polyhedron $\mathcal{P}$ that would complete the extended formulation.

We can make a similar analysis of necessary conditions in the case of a mixed binary convex quadratic representable set $S \subseteq \mathbb{R}^n$. It follows from Theorem 4 that $S$ must be the union of convex regions $R_1, \dots, R_k$ where each $R_i$ is a continuous convex quadratic representable set. It can be checked that each of the regions $R_i$ must have the same set of recession directions. However, these necessary conditions are not sufficient.

**An example of a set that is not representable.** Consider the set $S \subseteq \mathbb{R}^2$ illustrated in Figure 3 below and described by $S = S_1 \cup S_2$ where $S_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y \leq 0,\ x \geq 1\}$ and $S_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y \leq 0,\ x \leq -1\}$. Then $S$ is the union of two continuous convex quadratic representable sets with the same recession cone, and thus meets the two obvious necessary conditions described above. We will show however, that $S$ is *not* mixed binary convex quadratic representable. In order to do so, we will derive a stronger necessary condition for mixed binary convex quadratic representable sets. †

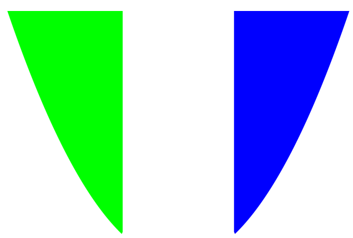

Figure 3: A set that is *not* mixed binary convex quadratic representable

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Let $a \in \mathbb{R}^n$ be a nonzero vector. We say that $a$ is an *unbounded linear objective of* $C$ if $\max\{a^\top x \mid x \in C\} = +\infty$. We can now establish the following proposition.

**Proposition 2.** *Let $\mathcal{Q}$ be a convex quadratic region in $\mathbb{R}^n$ described by*

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Q x + (Qw + v)^\top x + g \leq 0\},$$

*where $v \in \ker(Q)$ and $v \neq 0$. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron. Assume that $\mathcal{Q} \cap \mathcal{P}$ is nonempty. Then $a \in \mathbb{R}^n$ is an unbounded linear objective of $\mathcal{Q} \cap \mathcal{P}$ if and only if either*

*(a) there exists $r \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{Q})$ such that $a^\top r > 0$; or*

*(b) there exist both $r \in \text{rec}(\mathcal{P}) \cap \text{relint}(\text{rec}(\mathcal{Q}))$ such that $a^\top r \geq 0$ and $s \in \text{rec}(\mathcal{P})$ such that $a^\top s > 0$.*

*Proof.* We first note that by Observation 2, $\text{relint}(\text{rec}(\mathcal{Q})) = \{x \in \mathbb{R}^n \mid Qx = 0,\ v^\top x < 0\}$.

Assume first that there exists $r \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{Q})$ such that $a^\top r > 0$. Fix $M \geq 0$. We show how to find a point in $\mathcal{Q} \cap \mathcal{P}$ with objective value at least $M$. Let $\bar{x} \in \mathcal{Q} \cap \mathcal{P}$. There exists $\lambda \geq 0$ such that $\bar{x} + \lambda r \in \mathcal{Q} \cap \mathcal{P}$ and $a^\top(\bar{x} + \lambda r) \geq M$. Thus, $a$ is an unbounded linear direction of $\mathcal{Q} \cap \mathcal{P}$.

Suppose there exist $r \in \text{rec}(\mathcal{P}) \cap \text{relint}(\text{rec}(\mathcal{Q}))$ such that $a^\top r \geq 0$ and $s \in \text{rec}(\mathcal{P})$ such that $a^\top s > 0$. Fix $M \geq 0$. We show how to find a point in $\mathcal{Q} \cap \mathcal{P}$ with objective value at least $M$. Let $\bar{x} \in \mathcal{Q} \cap \mathcal{P}$. Now, we may assume that $M \geq a^\top \bar{x}$ else we are done. Set $\gamma = \frac{M - a^\top \bar{x}}{a^\top s}$ and $y = \bar{x} + \gamma s$. Then, since $v^\top r < 0$, for any $\lambda \geq \max\{-\frac{y^\top Q y + (Qw+v)^\top y + g}{v^\top r}, 0\}$, we have $a^\top(y + \lambda r) \geq a^\top y = M$. It now suffices to show that $y + \lambda r \in \mathcal{Q} \cap \mathcal{P}$. Since $r, s \in \text{rec}(\mathcal{P})$, clearly $y + \lambda r = \bar{x} + \gamma s + \lambda r \in \mathcal{P}$. Now since $Qr = 0$, we have

$$(y+\lambda r)^\top Q(y+\lambda r) + (Qw+v)^\top (y+\lambda r) + g = \lambda v^\top r + y^\top Q y + (Qw+v)^\top y + g \leq 0,$$

by choice of $\lambda$. Thus, $y + \lambda r \in \mathcal{Q} \cap \mathcal{P}$ and $a$ is an unbounded linear objective of $\mathcal{Q} \cap \mathcal{P}$.

We prove the reverse direction by induction on $\dim(\text{lin}(\mathcal{Q}))$. Suppose that $a$ is an unbounded linear objective of $\mathcal{Q} \cap \mathcal{P}$. Then there exists a sequence $\{x^k\}$ in $\mathcal{Q} \cap \mathcal{P}$ such that $a^\top x^k \to +\infty$. Let $w^k = \frac{x^k}{||x^k||}$. Then $\{w^k\}$ is a bounded sequence and therefore must have a convergent subsequence. Suppose $\bar{w}$ is a limit point of this sequence. Then $\bar{w}$ is a unit vector, satisfies $a^\top \bar{w} \geq 0$, and it is a fact of convex analysis that $\bar{w} \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{Q})$.

If $\bar{w}$ satisfies $a^\top \bar{w} > 0$ then we have met condition (a), and we are done. Thus, we may assume that $a^\top \bar{w} = 0$. Since $a$ is an unbounded linear objective of $\mathcal{P}$, it follows from the Minkowski-Weyl decomposition theorem that there exists $s \in \text{rec}(\mathcal{P})$ such that $a^\top s > 0$. If $\bar{w}$ satisfies $v^\top \bar{w} < 0$ then $\bar{w} \in \text{relint}(\text{rec}(\mathcal{Q}))$ and we are done.

In the base case, $\dim(\text{lin}(\mathcal{Q})) = 0$, we have $\text{rec}(\mathcal{Q}) = \{\lambda v \mid \lambda \leq 0\}$ and since $\bar{w} \neq 0$ it follows that $v^\top \bar{w} < 0$.

In order to prove the inductive step we assume that either condition (a) or (b) holds for an unbounded linear objective provided $\dim(\text{lin}(\mathcal{Q})) < k$. Assume $\dim(\text{lin}(\mathcal{Q})) = k$. By the same construction as before, we either meet condition (a) or (b) or have a vector $\bar{w} \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{Q})$ satisfying $a^\top \bar{w} = 0$ and $v^\top \bar{w} = 0$.

It remains to find $r \in \text{rec}(\mathcal{P}) \cap \text{relint}(\text{rec}(\mathcal{Q}))$ satisfying $a^\top r \geq 0$. We note that by Observation 2, we have $\bar{w} \in \text{lin}(\mathcal{Q})$. Consider the projection of $\mathcal{Q} \cap \mathcal{P}$ and the vector $a$ onto the orthogonal complement of $\text{Span}(\{\bar{w}\})$. Let $\mathcal{Q}'$ denote the projection of $\mathcal{Q}$, $\mathcal{P}'$ the projection of $\mathcal{P}$, and $a'$ the projection of $a$. Since $\bar{w} \in \text{lin}(\mathcal{Q})$ and $a^\top \bar{w} = 0$ we have that $a'$ is unbounded linear objective of $\mathcal{Q}' \cap \mathcal{P}'$ and $\dim(\text{lin}(\mathcal{Q}')) = k - 1$. We can now apply the induction hypothesis to obtain either a vector $r' \in \text{rec}(\mathcal{P}') \cap \text{rec}(\mathcal{Q}')$ satisfying $a'^\top r' > 0$ or two vectors $u' \in \text{rec}(\mathcal{P}') \cap \text{relint}(\text{rec}(\mathcal{Q}'))$ satisfying $a'^\top u' \geq 0$ and $s' \in \text{rec}(\mathcal{P}')$ satisfying $a'^\top s' > 0$. We claim that by lifting the vectors $r', u'$, and $s'$ back to the original space, we can obtain vectors satisfying either (a) or (b) for the initial region $\mathcal{Q} \cap \mathcal{P}$.

Assume first that there exists $r' \in \text{rec}(\mathcal{P}') \cap \text{rec}(\mathcal{Q}')$ satisfying $a'^\top r' > 0$. Since $r' \in \text{rec}(\mathcal{P}')$ there exists $r \in \text{rec}(\mathcal{P})$ that projects down to $r'$. In particular,

$r = r' + \alpha\bar{w}$ for some $\alpha \in \mathbb{R}$. It follows that $r \in \operatorname{rec}(\mathcal{Q})$ and $a^\top r > 0$ so that condition (a) is met.

Assume now that there exist two vectors $u' \in \operatorname{rec}(\mathcal{P}') \cap \operatorname{relint}(\operatorname{rec}(\mathcal{Q}'))$ satisfying $a'^\top u' \geq 0$ and $s' \in \operatorname{rec}(\mathcal{P}')$ satisfying $a'^\top s' > 0$. Again, there exists $u \in \operatorname{rec}(\mathcal{P})$ such that $u = u' + \alpha\bar{w}$ for some $\alpha \in \mathbb{R}$. Since $\bar{w} \in \operatorname{lin}(\mathcal{Q})$ and $a^\top \bar{w} = 0$ , it follows that $u \in \operatorname{rec}(\mathcal{P}) \cap \operatorname{relint}(\operatorname{rec}(\mathcal{Q}))$ and $a^\top u \geq 0$. Similarly, there exists $s \in \operatorname{rec}(\mathcal{P})$ that projects down to $s'$. Then $a^\top s > 0$ and condition (b) is met. □

A description of unbounded linear objectives for convex quadratic regions, with $v \neq 0$, can be obtained by considering Proposition 2 when $\mathcal{P} = \mathbb{R}^n$. In this case, $a$ is an unbounded linear objective of $\mathcal{Q}$ if and only if there exists $r \in \operatorname{relint}(\operatorname{rec}(\mathcal{Q}))$ such that $a^\top r \geq 0$.

We note that a similar characterization of bounded linear objectives holds when $v = 0$, i.e., when $\mathcal{Q}$ is an ellipsoidal region, see [5, 4] for more details.

**Proposition 3.** *Let $\mathcal{E} \subseteq \mathbb{R}^n$ be an ellipsoidal region and $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron. Assume that $\mathcal{E} \cap \mathcal{P}$ is nonempty. Then $a \in \mathbb{R}^n$ is an unbounded linear objective of $\mathcal{E}\cap\mathcal{P}$ if and only if there exists $r \in \operatorname{rec}(\mathcal{P})\cap\operatorname{rec}(\mathcal{E})$ such that $a^\top r > 0$.*

*Proof.* We first note that by the proof of Claim 2 in [5], we have $\mathcal{E}\cap\mathcal{P} = B+\mathcal{C}$ for a bounded set $B \subseteq \mathbb{R}^n$ and a polyhedral cone $\mathcal{C}$. Moreover, by Observations 2 and 3 in [5], $\mathcal{C}$ is the polyhedral cone $\operatorname{rec}(\mathcal{P}) \cap \operatorname{rec}(\mathcal{E})$. Assume now that $a$ is an unbounded linear objective of $\mathcal{E} \cap \mathcal{P}$. Since $B$ is a bounded set, there exists $r \in \operatorname{rec}(\mathcal{P}) \cap \operatorname{rec}(\mathcal{E})$ such that $a^\top r > 0$. Assume now that there exists $r \in \operatorname{rec}(\mathcal{P}) \cap \operatorname{rec}(\mathcal{E})$ satisfying $a^\top r > 0$. Fix $\bar{x} \in \mathcal{E} \cap \mathcal{P}$ and $M \geq 0$. Since $a^\top r > 0$, there exists $\lambda \geq 0$ such that $\bar{x} + \lambda r \in \mathcal{E} \cap \mathcal{P}$ and $a^\top(\bar{x} + \lambda r) \geq M$. □

Again, a description of unbounded linear objectives for ellipsoidal regions can be recovered by considering Proposition 3 when $\mathcal{P} = \mathbb{R}^n$. In this case, $a$ is an unbounded linear objective of $\mathcal{E}$ if and only if there exists $r \in \operatorname{rec}(\mathcal{E})$ such that $a^\top r > 0$.

Together Propositions 2 and 3 describe the set of unbounded linear objectives of sets that are the intersection of a convex quadratic region and a polyhedron. The following corollary to Propositions 2 and 3 establishes a new necessary condition for mixed binary convex quadratic representable sets.

**Corollary 1.** *Let $S \subseteq \mathbb{R}^n$ be a mixed binary convex quadratic representable set. Then there exist continuous convex quadratic representable sets $R_1, \ldots, R_k \subseteq \mathbb{R}^n$ each with the same set of unbounded linear objectives such that $S = \cup_{i=1}^k R_i$.*

*Proof.* Since $S$ is mixed binary convex quadratic representable, there exists a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p+q}$ and a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p+q}$ such that

$$S = \operatorname{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0,1\}^q)).$$

Take $R_1, \ldots, R_k$ to be the nonempty regions among $\operatorname{proj}_n((\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}})$ for $\bar{z} \in \{0,1\}^q$. Then clearly, each $R_i$ is continuous convex quadratic representable,

$S = \cup_{i=1}^k R_i$, and it remains to show that each $R_i$ has the same set of unbounded linear objectives.

Now, by Propositions 2 and 3 the set of unbounded linear objectives of a non-empty set $\mathcal{Q}' \cap \mathcal{P}'$ depends only on the recession cone of $\mathcal{Q}'$ and the recession cone of $\mathcal{P}'$. We now apply Lemma 1 to observe that each region $(\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}}$ has shape of $\mathcal{Q}|_{z=\bar{z}}$ and recession cone of $\mathcal{P}|_{z=\bar{z}}$ independent of choice of $\bar{z} \in \{0,1\}^q$. It then follows by Observation 2, that the recession cone of $\mathcal{Q}|_{z=\bar{z}}$ is independent of choice of $\bar{z} \in \{0,1\}^q$. In particular, each nonempty $(\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}}$ has the same set of unbounded linear objectives. It follows that their projections, $R_1, \ldots, R_k$ have the same set of unbounded linear objectives as well. □

**An example of a set that is not representable (cont.).** Assume then that $S$ is mixed binary convex quadratic representable. Then, by Corollary 1, $S$ decomposes into a union of regions $R_1, \ldots, R_k$ each with the same set of unbounded linear objectives. However, the two regions $S_1$ and $S_2$ do not have the same set of unbounded linear objectives. In particular, $(1,0)^\top$ must be an unbounded linear objective for at least one $R_i$ contained in $S_1$. However, $\max\{x \mid x \in S_2\} = -1$ which implies that $(1,0)^\top$ is not an unbounded linear objective for some $R_j$, a contradiction. It follows that $S$ is not mixed binary convex quadratic representable. †

We note that Corollary 1 imposes a stronger necessary condition on mixed binary convex quadratic representable sets than our initial observation provides. It is unclear whether stronger necessary conditions are required. Thus, we are left to consider the following question.

**Question 2.** *Let $S \subseteq \mathbb{R}^n$. Is it true that $S$ is mixed binary convex quadratic representable if and only if there exist continuous convex quadratic representable sets $R_1, \ldots, R_k \subseteq \mathbb{R}^n$ each with the same set of unbounded linear objectives such that $S = \cup_{i=1}^k R_i$ ?*

As for Question 1, in order to show that Question 2 is true, the main difficulty is in finding a suitable extended formulation for the given set $S$. This is due to the fact that the extended formulations pervasive throughout disjunctive programming fail in the presence of nonlinear constraints. While such extended formulations can be altered to behave nicely under certain conditions, e.g., when $S$ is bounded, it seems that entirely different formulations must be found for the general case.

If we were to show that the questions were false, we should search for strictly stronger necessary conditions satisfied by the respective classes of representable sets. The algebraic characterizations found in Sections 4 and 5 provide a solid foundation for this search. In particular, there is still much to explore in the projection procedure described in Section 4. At the current moment however, it is unclear what further sort of geometric conditions are implied by the algebraic characterizations.

An interesting future work would be exploring whether imposing stronger conditions on a given set $S \subseteq \mathbb{R}^n$ would lead to a readily constructible extended formulation. In particular, can we find certain classes of mixed binary convex quadratic representable sets for which we can provide explicit extended formulations?

The notion of unbounded linear objective is quite similar to the notion of *thin convex sets* explored in [6]. Let $K \subseteq \mathbb{R}^n$ be a closed convex set. A set $K$ is *thin* if the following holds for all $a \in \mathbb{R}^n$: $\max\{a^\top x \mid x \in K\} = +\infty$ if and only if there exists $r \in \text{rec}(K)$ such that $a^\top r > 0$. We conclude this paper by characterizing which convex quadratic regions are thin.

**Proposition 4.** *Let $\mathcal{Q} \subseteq \mathbb{R}^n$ be a convex quadratic region. Then $\mathcal{Q}$ is thin if and only if $\mathcal{Q}$ is either an ellipsoidal region or a half-space.*

*Proof.* Suppose that $\mathcal{Q}$ is described by

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Qx + (Qw + v)^\top x + g \leq 0\}$$

where $v \in \ker(Q)$.

Assume first that $\mathcal{Q}$ is either an ellipsoidal region or a half-space. Since a half-space is clearly thin, we may assume that $\mathcal{Q}$ is an ellipsoidal region. Then by the characterization of unbounded linear objectives of $\mathcal{Q}$ following Proposition 3 $\mathcal{Q}$ is thin.

Assume now that $\mathcal{Q}$ is neither an ellipsoidal region nor a half-space. This implies that $Q \neq 0$ and $v \neq 0$. Then by the characterization of unbounded linear objectives of $\mathcal{Q}$ following Proposition 2, any nonzero vector $a$ in the orthogonal complement of $\ker(\mathcal{Q})$ is an unbounded linear objective of $\mathcal{Q}$. Any such vector $a$ is orthogonal to all vectors in $\text{rec}(\mathcal{Q})$ and thus $\mathcal{Q}$ is not thin. □

In [6], the authors show that if a closed convex set $K \subseteq \mathbb{R}^n$ with $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ is not thin, then the region $\text{conv}(K \cap \mathbb{Z}^n)$ is not a polyhedron. Thus, for a general convex quadratic set $\mathcal{Q} \subseteq \mathbb{R}^n$ the region $\text{conv}(\mathcal{Q} \cap \mathbb{Z}^n)$ is not a polyhedron. The lack of a succinct description of the points in $\mathcal{Q} \cap \mathbb{Z}^n$ is one of the reasons we do not investigate extended integer variables in this work. A possible future work could consider this more general setting.

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