Ellipsoidal Mixed-Integer Representability

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Abstract

Representability results for mixed-integer linear systems play a fundamental role in optimization since they give geometric characterizations of the feasible sets that can be formulated by mixed-integer linear programming. We consider a natural extension of mixed-integer linear systems obtained by adding just one ellipsoidal inequality. The set of points that can be described, possibly using additional variables, by these systems are called ellipsoidal mixed-integer representable. In this work, we give geometric conditions that characterize ellipsoidal mixed-integer representable sets.

Key words: mixed-integer programming; quadratic programming; representability; ellipsoidal constraints

1 Introduction

The theory of representability starts with a paper of Dantzig [2] and studies one fundamental question: Given a system of algebraic constraints of a specified form, which subsets of $\mathbb{R}^n$ can be represented in this way, possibly using additional variables? Several researchers have investigated representability questions (see, e.g., [5, 6, 15]), and a systematic study for mixed-integer linear systems is mainly due to Meyer and Jeroslow (see [10, 11, 7, 12, 9, 8]).

Since projections of polyhedra are polyhedral (see [14]), the sets representable by systems of linear inequalities are polyhedra. More formally, a set $S \subseteq \mathbb{R}^n$ is representable as the orthogonal projection onto $\mathbb{R}^n$ of the solution set of a linear system

$$\begin{aligned} Dw &\le d \\ w &\in \mathbb{R}^{n+p} \end{aligned}$$

if and only if S is a polyhedron.

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If we also allow binary or integer extended variables, then geometric characterizations have been given by Jeroslow and Lowe [8, 9]. A set $S \subseteq \mathbb{R}^n$ is representable as the orthogonal projection onto $\mathbb{R}^n$ of the solution set of a mixed-binary linear system

$$\begin{aligned} &Dw \le d \\ &w \in \mathbb{R}^{n+p} \times \{0,1\}^q \end{aligned}$$

if and only if S is the union of a finite number of polyhedra with the same recession directions.

A set $S \subseteq \mathbb{R}^n$ is representable as the orthogonal projection onto $\mathbb{R}^n$ of the solution set of a *rational* mixed-integer linear system

$$\begin{aligned} &Dw \le d \\ &w \in \mathbb{R}^{n+p} \times \mathbb{Z}^q \end{aligned}$$

if and only if S is the union of a finite number of rational polytopes plus the set of nonnegative integer combinations of finitely many integral vectors.

We are interested in giving representability results for mixed-integer sets defined not only by linear inequalities, but also by quadratic inequalities of the form $(x-c)^\top Q(x-c) \le \gamma$, where Q is a positive semidefinite matrix. Inequalities of this type are called *ellipsoidal inequalities*. More precisely, we say that a set $\mathcal{E} \subseteq \mathbb{R}^n$ is an *ellipsoidal region* in $\mathbb{R}^n$ if there exists an $n \times n$ matrix $Q \succeq 0$ (i.e., Q is positive semidefinite), a vector $c \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$, such that

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid (x-c)^\top Q(x-c) \le \gamma\}.$$

We say that an ellipsoidal region $\mathcal{E}$ in $\mathbb{R}^n$ is *rational* if we can take the data defining $\mathcal{E}$, namely Q, c, and γ to be rational.

Ellipsoidal inequalities arise in many practical applications. As an example, many real-life quantities are normally distributed; and for a normal distribution, a natural confidence set, containing the vast majority of the objects, is an ellipsoidal region. See, e.g., [16] for other applications of ellipsoidal inequalities.

A characterization of sets representable by an arbitrary number of ellipsoidal inequalities seems to be currently out of reach. We illustrate this difficulty with a routine example. Consider the set $S \subseteq \mathbb{R}^2$ defined as the projection of

$$\{(x_1, x_2, y) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + y^2 \le 1,\ (x_1 + y - 1)^2 + x_2^2 \le 1\}$$

onto the space defined by the coordinates (x_1, x_2). The region S is depicted in Figure 1.

It can be shown that S is not a basic semialgebraic set. Moreover, the highlighted component of the boundary of S is described by the vanishing of a degree four polynomial. It can be checked that this component of the boundary cannot be described by any polynomials of degree three or less. Thus, degree four polynomials are essential to the description of S. These phenomena make it difficult to obtain complete characterization theorems similar to those of Jeroslow

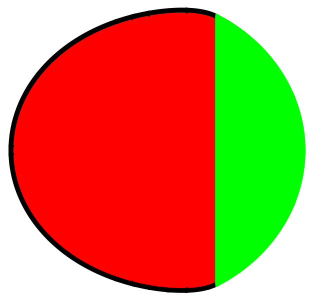


Figure 1: The projection of the intersection of two ellipsoidal regions

and Lowe. As a consequence, in this work we will focus on understanding the expressing power of just one ellipsoidal inequality.

Formally, we say that a set $S \subseteq \mathbb{R}^n$ is *binary ellipsoidal mixed-integer (binary EMI) representable* if it can be obtained as the orthogonal projection onto $\mathbb{R}^n$ of the solution set of a system of the form

$$\begin{aligned} &Dw \leq d \\ &(w-c)^\top Q(w-c) \leq \gamma \\ &w \in \mathbb{R}^{n+p} \times \{0,1\}^q, \end{aligned} \tag{1}$$

where Q is positive semidefinite.

In the general case where extended variables can be mixed-integer, we have a similar definition. We say that a set $S \subseteq \mathbb{R}^n$ is *ellipsoidal mixed-integer (EMI) representable* if it can be obtained as the orthogonal projection onto $\mathbb{R}^n$ of the solution set of a system of the form

$$\begin{aligned} &Dw \leq d \\ &(w-c)^\top Q(w-c) \leq \gamma \\ &w \in \mathbb{R}^{n+p} \times \mathbb{Z}^q, \end{aligned} \tag{2}$$

where Q is positive semidefinite. We say that a set $S \subseteq \mathbb{R}^n$ is *rational EMI-representable* if the data D, Q, d, c, γ can be chosen to be rational. We note that any binary EMI-representable set is EMI-representable.

There is a strong connection between EMI-representable sets and Mixed-Integer Quadratic Programming (MIQP). In a MIQP problem we aim at minimizing a quadratic function over mixed-integer points in a polyhedron. Since every MIQP with bounded objective admits an optimal solution of polynomial size (see [3]), any MIQP is equivalent to a polynomial number of MIQP feasibility problems. If the objective quadratic is ellipsoidal, then each feasibility problem is over a set of the form (2).

Our main results are the following geometric characterizations of EMI-representable sets. In the binary case, we have the following theorem which was first presented in the IPCO version of this paper [4].

Theorem 1. *A set $S \subseteq \mathbb{R}^n$ is binary EMI-representable if and only if there exist ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that*

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}. \tag{3}$$

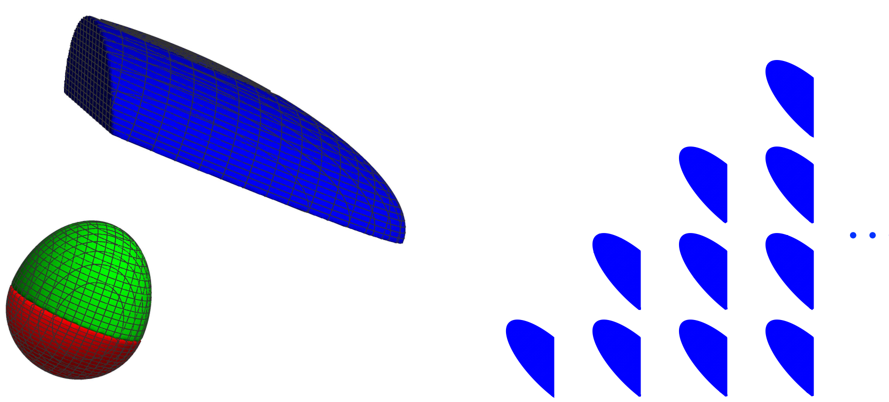
In the rational mixed-integer case, we have the following result. Given $R \subseteq \mathbb{R}^n$, we define the integer cone of R to be

$$\text{int.cone}(R) := \Big\{ \sum_{i=1}^{t} \mu_i r^i \;\Big|\; r^i \in R,\ \mu_i \in \mathbb{Z}_{\geq 0},\ i = 1, \ldots, t \Big\}.$$

Theorem 2. *A set $S \subseteq \mathbb{R}^n$ is rational EMI-representable if and only if there exist rational ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, rational polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and integral vectors $r^i \in \mathbb{Z}^n$, $i = 1, \ldots, t$ such that*

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \text{int.cone}\{r^1, \ldots, r^t\}. \tag{4}$$

An example of a binary EMI-representable set is given in Figure 2(a) while an example of an EMI-representable set is given in Figure 2(b). Note that the second set is not binary EMI-representable as it is the disjoint union of an infinite number of convex regions.



(a) A binary EMI-representable set in $\mathbb{R}^3$

(b) An EMI-representable set in $\mathbb{R}^2$

Figure 2: Examples of EMI-representable sets

The presence of rational data in Theorem 2 is essential to the development of a meaningful statement. Even in the pure integer linear case, irrational data may cause complications. Consider the integer set $S = \{(x_1, x_2) \in \mathbb{Z}^2_{\geq 0} \mid x_2 \leq \sqrt{2}\, x_1\}$. It is well known that S cannot be represented as the Minkowski sum of a finite set and the integer cone of a finite number of integral vectors.

Both directions of Theorem 1 and Theorem 2 have geometric implications. Since each set S of the form (3) or (4) can be obtained as the projection of a set described by a system (1) or (2) this means that the k ellipsoidal regions $\mathcal{E}_i$ can be expressed with just one ellipsoidal inequality in a higher dimension. We prove this direction of the theorems by explicitly giving extended formulations for the sets S.

The other direction of Theorem 1 and Theorem 2 states that the projection of each system (1) or (2) onto $\mathbb{R}^n$ is a set of the form (3) or (4). An important ingredient of both proofs is showing that the projection of a set

$$\mathcal{E} \cap \mathcal{P} := \{w \in \mathbb{R}^{n+1} \mid Dw \leq d,\ (w-c)^\top Q(w-c) \leq \gamma\}$$

onto $\mathbb{R}^n$ is a set of the form (3). In order to do so, we introduce the key concept of a *shadowing hyperplane*. This hyperplane allows us to split the ellipsoidal region into two 'parts'. In turn, this allows us to compute the projection of $\mathcal{E} \cap \mathcal{P}$ by computing a finite number of projections of $\mathcal{E}$ intersected with a hyperplane. This will show that the projection of $\mathcal{E} \cap \mathcal{P}$ is the union of a finite number of regions that are the intersection of a polyhedron and one nonlinear inequality, which we will prove to be ellipsoidal.

The remainder of this paper is organized as follows. In Section 2, we provide a number of results relating to the intersection of an ellipsoidal region with a polyhedron and the projections of such regions. In Section 3, we prove our main results Theorem 1 and Theorem 2.

Notation. In the remainder of the paper we will use the following notation. Given a set $E \subseteq \mathbb{R}^n \times \mathbb{R}^p$ and a vector $\bar{y} \in \mathbb{R}^p$, we define the *$\bar{y}$-restriction* of E as

$$E|_{y=\bar{y}} = \{x \in \mathbb{R}^n \mid (x, \bar{y}) \in E\}.$$

Note that $E|_{y=\bar{y}}$ geometrically consists of the intersection of E with coordinate hyperplanes. Sometimes we will need to consider $E|_{y=\bar{y}}$ in the original space $\mathbb{R}^n \times \mathbb{R}^p$, thus we also define

$$\tilde{E}|_{y=\bar{y}} = \{(x, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^p \mid (x, \bar{y}) \in E\}.$$

We may also need to fix a subset of the y-coordinates $y_1 = \bar{y}_1, \ldots, y_k = \bar{y}_k$ at one time. In such a case we simply write $E|_{y_1=\bar{y}_1,\ldots,y_k=\bar{y}_k}$ and $\tilde{E}|_{y_1=\bar{y}_1,\ldots,y_k=\bar{y}_k}$.

Given a set $E \subseteq \mathbb{R}^n$, and a positive integer $k \leq n$, we will denote by $\operatorname{proj}_k(E)$ the orthogonal projection of E onto its first k coordinates. Formally,

$$\operatorname{proj}_k(E) = \{x \in \mathbb{R}^k \mid \exists y \in \mathbb{R}^{n-k} \text{ with } (x, y) \in E\}.$$

We note that $\text{proj}_k : \mathbb{R}^n \to \mathbb{R}^k$ is a linear transformation, and thus respects vector addition, i.e., Minkowski sums. We also denote by $\text{span}(E)$ the linear space generated by E and by $\text{cone}(E)$ the cone generated by E.

Given a nonempty convex set $E \subseteq \mathbb{R}^n$ we denote by $\text{rec}(E)$ the recession cone of E, namely the set of vectors $r \in \mathbb{R}^n$ such that for any $\lambda > 0$ and $x \in E$ we have $x + \lambda r \in E$. We note that nearly all of the sets we consider in this paper are closed, in which case $\text{rec}(E)$ coincides with the set of recession directions at *any* point of E, see Theorem 8.3 in [13]. We also denote by $\text{lin}(E)$ the lineality space of E.

Given a matrix A we denote by $\text{range}(A)$ the range of A and by $\ker(A)$ the kernel of A. If A is positive semidefinite, we write $A \succeq 0$. This implies that A is symmetric.

2 Ellipsoidal Regions and Hyperplanes

In this section we formally define ellipsoidal regions. These regions will appear throughout our study of representability. We will prove a few results on the intersection of ellipsoidal regions with half-spaces as well as their projections. These results will be necessary for our proofs of Theorem 1 and Theorem 2.

We say that a set $\mathcal{E}$ is an *ellipsoidal region* in $\mathbb{R}^n$ if there exists an $n \times n$ matrix $Q \succeq 0$, a vector $c \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$, such that

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid (x - c)^\top Q(x - c) \le \gamma\}.$$

We note that if $Q \succ 0$ (i.e., Q is positive definite) and $\gamma > 0$, then $\mathcal{E}$ is an *ellipsoid*, i.e., the image of the unit ball $\mathcal{B} = \{x \in \mathbb{R}^n \mid ||x||_2 \le 1\}$ under an invertible affine transformation.

The following observation is well-known, and we give a proof for completeness.

Observation 1. *Let $q(x) = x^\top Qx + b^\top x$ be a quadratic function on $\mathbb{R}^n$ with Q a positive semidefinite matrix. Then $q(x)$ has a minimum on $\mathbb{R}^n$ if and only if b is in the range of Q.*

Proof. Assume $b \notin \text{range}(Q)$. Then since Q is symmetric, we can write $b = Qr + c$ with $Qc = 0$ and $c \neq 0$. Consider the line $x(t) = -tc$ for $t \in \mathbb{R}$. Then we have

$$q(x(t)) = b^\top x(t) = -tc^\top c.$$

Since $c \neq 0$, we see that $q(x(t)) \to -\infty$ as $t \to +\infty$. Thus, $q(x)$ has no minimum on $\mathbb{R}^n$.

Assume there exists $r \in \mathbb{R}^n$ such that $\frac{1}{2}b = Qr$. Then

$$q(x) = (x + r)^\top Q(x + r) - r^\top Qr$$

and $q(x)$ has a minimum at any $\bar{x}$ such that $\bar{x} + r \in \ker(Q)$. In particular, $-r$ is a minimizer and $q(-r) = -r^\top Qr$ is the optimal value. □ □

The following lemma shows that ellipsoidal regions are closed under intersections with coordinate hyperplanes. This is equivalent to fixing a number of variables.

Lemma 1. *Let $\mathcal{E}$ be an ellipsoidal region in $\mathbb{R}^n \times \mathbb{R}^p$. Then for any $\bar{y} \in \mathbb{R}^p$, the set $\mathcal{E}|_{y=\bar{y}}$ is an ellipsoidal region in $\mathbb{R}^n$.*

Proof. Let $\mathcal{E} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid q(x, y) \leq \gamma\}$, where $q(x, y)$ is the quadratic polynomial

$$q(x,y) = \begin{pmatrix} x-c \\ y-c' \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & Q' \end{pmatrix} \begin{pmatrix} x-c \\ y-c' \end{pmatrix}.$$

For any fixed $\bar{y} \in \mathbb{R}^p$, since Q is positive semidefinite it suffices to show there exists $c_{\bar{y}} \in \mathbb{R}^n$ and $\gamma_{\bar{y}} \in \mathbb{R}$ such that

$$\mathcal{E}|_{y=\bar{y}} = \{x \in \mathbb{R}^n \mid (x - c_{\bar{y}})^\top Q(x - c_{\bar{y}}) \leq \gamma_{\bar{y}}\}. \tag{5}$$

Let $\bar{y} \in \mathbb{R}^p$. Since $q(x, y)$ has a minimum on $\mathbb{R}^n \times \mathbb{R}^p$ by Observation 1, the quadratic function

$$q(x, \bar{y}) = (x - c)^\top Q(x - c) + 2(\bar{y} - c')^\top R^\top(x - c) + (\bar{y} - c')^\top Q'(\bar{y} - c'),$$

has a minimum on $\mathbb{R}^n$ as it is bounded from below. By Observation 1, $R(\bar{y}-c') \in \text{range}(Q)$, and so there exists $\bar{x} \in \mathbb{R}^n$ such that $Q\bar{x} = R(\bar{y} - c')$. Then (5) is satisfied with $c_{\bar{y}} := c - \bar{x}$ and $\gamma_{\bar{y}} := \gamma + \bar{x}^\top Q\bar{x} - (\bar{y} - c')^\top Q'(\bar{y} - c')$. □ □

We now provide a lemma that describes the recession cone of a nonempty ellipsoidal region.

Lemma 2. *Let $\mathcal{E} = \{x \in \mathbb{R}^n \mid (x-c)^\top Q(x-c) \leq \gamma\}$ be a nonempty ellipsoidal region in $\mathbb{R}^n$. Then*

$$\text{rec}(\mathcal{E}) = \ker(Q) = \{x \in \mathbb{R}^n \mid x^\top Qx = 0\}.$$

Proof. We first show that $\text{rec}(\mathcal{E}) = \ker(Q)$. Since $\mathcal{E}$ is a closed convex set, $\text{rec}(\mathcal{E})$ is equal to the set of recession directions from any point $x \in \mathcal{E}$. Consider the point $c \in \mathcal{E}$. Then for any $r \in \ker(Q)$ and $\lambda > 0$ we have $c + \lambda r \in \mathcal{E}$ since $\lambda^2 r^\top Qr = 0 \leq \gamma$. Thus, $r \in \text{rec}(\mathcal{E})$. Assume now that $r \in \text{rec}(\mathcal{E})$. Let $Q = L^\top L$ be a Cholesky decomposition of Q. Then for any $\lambda > 0$ we have $\lambda^2 r^\top Qr = \lambda^2 ||Lr||^2 \leq \gamma$, which implies $Lr = 0$ and $r \in \ker(Q)$. Next we show that $\ker(Q) = \{x \in \mathbb{R}^n \mid x^\top Qx = 0\}$. Clearly, the kernel is contained in the right hand side. Suppose $r \in \mathbb{R}^n$ satisfies $r^\top Qr = 0$. Replacing Q with its Cholesky decomposition, we see that $||Lr||^2 = 0$. This implies $Lr = 0$, and thus $r \in \ker(Q)$. □ □

We are now ready to provide a geometric description of ellipsoidal regions. A consequence of this description is that any non-empty ellipsoidal region may be decomposed as the Minkowski sum of an ellipsoid and a linear space.

Lemma 3. *Let $\mathcal{E}$ be an ellipsoidal region in $\mathbb{R}^n$. Then* *(i)* $\mathcal{E} = \emptyset$*, or*

(ii) $\mathcal{E}$ *is an affine space, or*

(iii) There exists an integer $k \in \{0, \ldots, n-1\}$*, a* k*-dimensional linear space* $L \subseteq \mathbb{R}^n$*, and* k *distinct indices* $i_1, \ldots, i_k \in \{1, \ldots, n\}$ *such that the restriction*

$$\mathcal{E}|_{x_{i_1}=\bar{x}_{i_1},\ldots,x_{i_k}=\bar{x}_{i_k}}$$

is an ellipsoid in $\mathbb{R}^{n-k}$*, and*

$$\mathcal{E} = \tilde{\mathcal{E}}|_{x_{i_1}=\bar{x}_{i_1},\ldots,x_{i_k}=\bar{x}_{i_k}} + L.$$

Proof. Let $\mathcal{E} = \{x \in \mathbb{R}^n \mid (x-c)^\top Q(x-c) \leq \gamma\}$ where Q is a positive semidefinite matrix. If $\gamma < 0$, then $\mathcal{E} = \emptyset$ since Q is positive semidefinite. Thus, we may assume that $\gamma \geq 0$ and $\mathcal{E}$ is non-empty.

Now assume $\gamma = 0$. By Lemma 2, $x \in \mathcal{E}$ if and only if $x \in c + \ker(Q)$. Thus $\mathcal{E} = c + \ker(Q)$ is an affine space.

Assume now $\gamma > 0$. If Q is invertible then $\mathcal{E}$ is an ellipsoid and we are done, in this case $k = 0$. Thus, we may assume $L := \ker(Q)$ is nontrivial. Let $k = \dim(L)$ and $\mathcal{L} = \{l_1, \ldots, l_k\}$ be a basis for L. Note if $k = n$ then $\mathcal{E} = \mathbb{R}^n$, an affine space, and we are done. Thus, we may assume that $k < n$. Extend $\mathcal{L}$ to a basis $\mathcal{L}'$ of $\mathbb{R}^n$ by adding a subset of the standard basis vectors $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. Let $\mathcal{J} \subseteq \{1, \ldots, n\}$ be the set of indices j for which $e_j \in \mathcal{L}' - \mathcal{L}$, and suppose $\{i_1, \ldots, i_k\} = \{1, \ldots, n\} - \mathcal{J}$. Define

$$\mathcal{E}' := \mathcal{E}|_{x_{i_1}=0,\ldots,x_{i_k}=0} \text{ and } \tilde{\mathcal{E}}' := \tilde{\mathcal{E}}|_{x_{i_1}=0,\ldots,x_{i_k}=0}.$$

We now show $\mathcal{E} = \tilde{\mathcal{E}}' + L$. Since $\tilde{\mathcal{E}}' \subseteq \mathcal{E}$ and $\operatorname{rec}(\mathcal{E}) = L$, we clearly have $\tilde{\mathcal{E}}' + L \subseteq \mathcal{E}$. Let $v \in \mathcal{E}$. Expanding v in the basis $\mathcal{L}'$, we have for some $l \in L$ and scalars $\alpha_j \in \mathbb{R}$, that $v = l + \sum_{j \in \mathcal{J}} \alpha_j e_j$. Since $L = \operatorname{rec}(\mathcal{E})$ we have $v - l = \sum_{j \in \mathcal{J}} \alpha_j e_j \in \mathcal{E}'$ and $\mathcal{E} \subseteq \tilde{\mathcal{E}}' + L$.

By Lemma 1, $\mathcal{E}'$ is an ellipsoidal region in $\mathbb{R}^{n-k}$. Note first that $\mathcal{E}$ is full-dimensional in $\mathbb{R}^n$, i.e., has $n+1$ affinely independent vectors. This is immediate since $\gamma > 0$ and there exists a vector, namely $c \in \mathbb{R}^n$, for which the continuous function $(x-c)^\top Q(x-c)$ has value 0. If $\mathcal{E}'$ is unbounded, then $\mathcal{E}'$ has some recession direction outside of L which contradicts the fact that $\operatorname{rec}(\mathcal{E}) = L$. Moreover, since $\mathcal{E}'$ is bounded it follows from Lemma 2 that the matrix defining $\mathcal{E}'$ is invertible, and thus positive definite. Then $\mathcal{E}'$ is either an ellipsoid or a single point. Since $\mathcal{E} = \tilde{\mathcal{E}}' + L$ is full dimensional, and $\dim(L) = k < n$, $\mathcal{E}'$ cannot be a single point. □ □

We make the following remark about the proof of *(iii)* that will be used later. If one of the standard basis vectors of $\mathbb{R}^n$, say e_n, is not contained in L, then we may assume that x_n does not occur among the fixed variables $x_{i_1}, \ldots, x_{i_k}$. To see this, note that in completing the basis $\mathcal{L}$ of L to a basis of $\mathbb{R}^n$ we may first add the standard basis vector e_n to the set $\mathcal{L}$. In general, it is not possible to complete this procedure for more than one of the $e_j \notin L$ simultaneously.

It can be shown that an appropriate converse of Lemma 3 holds. This provides a complete geometric characterization of ellipsoidal regions. We use Lemma 3 to make the following observation that distinguishes ellipsoidal regions from general convex quadratic regions.

Observation 2. *Let $\mathcal{E}$ be a nonempty ellipsoidal region in $\mathbb{R}^n$. Then there exists a polyhedron $\mathcal{B} \subseteq \mathbb{R}^n$ such that $\mathcal{E} \subseteq \mathcal{B}$ and $\operatorname{rec}(\mathcal{B}) = \operatorname{rec}(\mathcal{E})$.*

Proof. By Lemma 3, $\mathcal{E}$ is either an affine space, or the Minkowski sum of an ellipsoid and a linear space. Since affine spaces are polyhedral, it suffices to assume that $\mathcal{E} = \tilde{\mathcal{E}}|_{x_{i_1}=\bar{x}_{i_1},\cdots,x_{i_k}=\bar{x}_{i_k}} + L$ for an ellipsoid $\mathcal{E}|_{x_{i_1}=\bar{x}_{i_1},\cdots,x_{i_k}=\bar{x}_{i_k}}$ in $\mathbb{R}^{n-k}$ and a linear space L. Since $\tilde{\mathcal{E}}|_{x_{i_1}=\bar{x}_{i_1},\cdots,x_{i_k}=\bar{x}_{i_k}}$ is a bounded set there exists a polytope $\tilde{\mathcal{B}}$ such that $\tilde{\mathcal{E}}|_{x_{i_1}=\bar{x}_{i_1},\cdots,x_{i_k}=\bar{x}_{i_k}} \subseteq \tilde{\mathcal{B}}$. Then the polyhedron defined by $\mathcal{B} := \tilde{\mathcal{B}} + L$ has the desired properties. □ □

The next observation gives a description of the recession cones that will be encountered in this paper.

Observation 3. *Let $\mathcal{P}$ be a polyhedron and $\mathcal{E}$ an ellipsoidal region in $\mathbb{R}^n$. Assume that $\mathcal{E} \cap \mathcal{P} \neq \emptyset$. Then $\operatorname{rec}(\mathcal{E} \cap \mathcal{P})$ is a polyhedral cone.*

Proof. We note that by Corollary 8.3.3 in [13] we have that $\operatorname{rec}(\mathcal{E}\cap\mathcal{P}) = \operatorname{rec}(\mathcal{E})\cap \operatorname{rec}(\mathcal{P})$. The set $\operatorname{rec}(\mathcal{P})$ is a polyhedral cone (see, e.g., [14]), and $\operatorname{rec}(\mathcal{E})$ is a linear space by Lemma 2. As a consequence $\operatorname{rec}(\mathcal{E} \cap \mathcal{P})$ is a polyhedral cone. □ □

The following lemma shows that to compute the projection of an ellipsoidal region $\mathcal{E}$ in $\mathbb{R}^n$, it suffices to consider the projection of $\mathcal{E} \cap H$ for a specific hyperplane $H \subseteq \mathbb{R}^n$. We will refer to such a hyperplane H as a *shadowing hyperplane*, as it contains enough information to completely describe the projection, or 'shadow', of $\mathcal{E}$. See Figure 3 for an illustration.

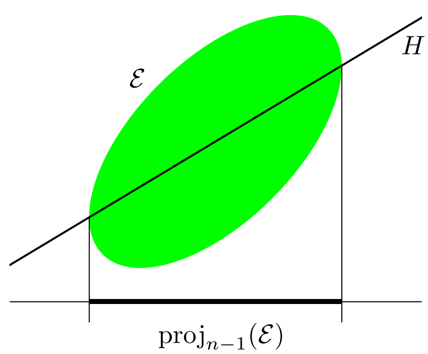


Figure 3: Illustration of a shadowing hyperplane

Lemma 4. *Let $\mathcal{E}$ be an ellipsoidal region in $\mathbb{R}^n$. Then there exists a hyperplane $H \subseteq \mathbb{R}^n$ with $e_n \notin \mathrm{lin}(H)$ such that*

$$\mathrm{proj}_{n-1}(\mathcal{E}) = \mathrm{proj}_{n-1}(\mathcal{E} \cap H).$$

Proof. If $\mathcal{E} = \emptyset$ the statement follows immediately. We assume then that $\mathcal{E} \neq \emptyset$. It clearly suffices to show that $\mathrm{proj}_{n-1}(\mathcal{E}) \subseteq \mathrm{proj}_{n-1}(\mathcal{E} \cap H)$. Let $\mathcal{E}$ be described by the ellipsoidal inequality $q(x) = (x-c)^\top Q(x-c) \leq \gamma$. We note that this inequality can be rearranged to $q(x) = x^\top Qx + b^\top x + d \leq 0$ for a specific vector $b \in \mathrm{range}(Q)$ and scalar $d \in \mathbb{R}$. Split the variable x into two pieces $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and write

$$Q = \begin{pmatrix} Q' & l \\ l^\top & a \end{pmatrix}$$

for an $(n-1) \times (n-1)$ matrix $Q' \succeq 0$ and scalar $a \geq 0$. After replacing b with (b', b_n) we can write

$$q(x', x_n) = ax_n^2 + (2l^\top x' + b_n)x_n + x'^\top Q'x' + b'^\top x' + d \leq 0.$$

Assume first that $a = 0$. Then $e_n \in \ker(Q)$ since $Q \succeq 0$. We claim that the hyperplane $H = \{x \in \mathbb{R}^n \mid x_n = 0\}$ has the desired property. For any $\bar{x} \in \mathrm{proj}_{n-1}(\mathcal{E})$, there exists λ such that $(\bar{x}, \lambda) \in \mathcal{E}$. Now by Lemma 2 we have $\pm e_n \in \mathrm{rec}(\mathcal{E})$. Then $(\bar{x}, 0) \in \mathcal{E} \cap H$ which implies $\bar{x} \in \mathrm{proj}_{n-1}(\mathcal{E} \cap H)$.

Assume now that $a > 0$. We claim that the hyperplane $H = \{x \in \mathbb{R}^n \mid 2ax_n + 2l^\top x' = -b_n\}$ has the desired property. Let $\bar{x} \in \mathrm{proj}_{n-1}(\mathcal{E})$. Then the univariate polynomial $q(\bar{x}, x_n)$ has real roots since $\bar{x} \in \mathrm{proj}_n(\mathcal{E})$ and $a > 0$. It follows from the quadratic formula that the midpoint on the line segment between the two roots say $(\bar{x}, \lambda)$ is in both $\mathcal{E}$ and H. □ □

We refer to the hyperplane H defined in Lemma 4 as the shadowing hyperplane of $\mathcal{E}$. The following proposition will be one of the main building blocks of both Theorem 1 and Theorem 2. It provides a geometric description of the projection of the intersection of an ellipsoidal region and a polyhedron.

Proposition 1. *Let $\mathcal{E} \subseteq \mathbb{R}^{n+p}$ be a nonempty ellipsoidal region and $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ be a nonempty polyhedron. Let $S = \mathrm{proj}_n(\mathcal{E} \cap \mathcal{P})$. Then there exist ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that*

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C},$$

where $\mathcal{C} = \mathrm{proj}_n(\mathrm{rec}(\mathcal{E}) \cap \mathrm{rec}(\mathcal{P}))$.

Proof. In the case that $\mathcal{E} \cap \mathcal{P} = \emptyset$, the statement follows immediately. We may now assume that $\mathcal{E} \cap \mathcal{P} \neq \emptyset$. In the first two claims we prove that it suffices to show that S has an equivalent, but simpler, decomposition.

Claim 1. *It suffices to find ellipsoidal regions* $\mathcal{E}_i \subseteq \mathbb{R}^n$*, polytopes* $\mathcal{P}_i \subseteq \mathbb{R}^n$*, and polyhedral cones* $\mathcal{C}_i \subseteq \mathbb{R}^n$*, for* $i = 1, \dots, k$*, that satisfy*

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i). \tag{6}$$

Proof of claim. We first show that $\mathrm{rec}(S) = \mathrm{proj}_n(\mathrm{rec}(\mathcal{E}) \cap \mathrm{rec}(\mathcal{P}))$, which by Observation 3 is a polyhedral cone. By definition, $\mathrm{rec}(S) = \mathrm{rec}(\mathrm{proj}_n(\mathcal{E} \cap \mathcal{P}))$. Then since the projection of a ray in $\mathcal{E} \cap \mathcal{P}$ is a ray in S, the containment of $\mathrm{proj}_n(\mathrm{rec}(\mathcal{E}) \cap \mathrm{rec}(\mathcal{P}))$ in $\mathrm{rec}(S)$ is clear. Let $r \in \mathrm{rec}(\mathrm{proj}_n(\mathcal{E} \cap \mathcal{P}))$. Consider a polyhedral approximation $\mathcal{B}$ of $\mathcal{E}$ as in Observation 2 such that $\mathcal{B} \subseteq \mathbb{R}^{n+p}$ is a polyhedron, $\mathcal{E} \subseteq \mathcal{B}$ and $\mathrm{rec}(\mathcal{E}) = \mathrm{rec}(\mathcal{B})$. Then clearly, $r \in \mathrm{rec}(\mathrm{proj}_n(\mathcal{B} \cap \mathcal{P}))$ and since $\mathcal{B} \cap \mathcal{P}$ is a polyhedron it is well known that $\mathrm{rec}(\mathrm{proj}_n(\mathcal{B} \cap \mathcal{P})) = \mathrm{proj}_n(\mathrm{rec}(\mathcal{B} \cap \mathcal{P}))$. Then by construction, $\mathrm{rec}(\mathcal{B} \cap \mathcal{P}) = \mathrm{rec}(\mathcal{B}) \cap \mathrm{rec}(\mathcal{P}) = \mathrm{rec}(\mathcal{E}) \cap \mathrm{rec}(\mathcal{P})$. Henceforth, we denote by $\mathcal{C}$ the polyhedral cone $\mathrm{rec}(S)$.

Assume we have $\mathcal{E}_i, \mathcal{P}_i$, and $\mathcal{C}_i$ that satisfy (6). Since $\mathcal{C} = \mathrm{rec}(S)$, for each $i = 1, \dots, k$ we have that $\mathcal{C}_i$ must be contained in $\mathcal{C}$. It follows that

$$S = S + C = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i) + \mathcal{C} = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}. \qquad \diamond$$

Claim 2. *It suffices to find ellipsoidal regions* $\mathcal{E}_i \subseteq \mathbb{R}^n$*, polyhedra* $\mathcal{P}_i \subseteq \mathbb{R}^n$*, for* $i = 1, \dots, k$*, that satisfy*

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i). \tag{7}$$

Proof of claim. Assume we have ellipsoidal regions $\mathcal{E}_i$ and nonempty polyhedra $\mathcal{P}_i$ that satisfy (7). Without loss of generality, we may assume that all of the $\mathcal{E}_i$ are nonempty. Consider a polyhedral approximation $\mathcal{B}_i$ of $\mathcal{E}_i$ as in Observation 2 such that $\mathcal{B}_i \subseteq \mathbb{R}^n$ is a polyhedron, $\mathcal{E}_i \subseteq \mathcal{B}_i$, and $\mathrm{rec}(\mathcal{E}_i) = \mathrm{rec}(\mathcal{B}_i)$. Then $\mathcal{B}_i \cap \mathcal{P}_i$ is a polyhedron and by the Minkowki-Weyl theorem can be decomposed as $\mathcal{R}_i + \mathcal{C}_i$ for a polytope $\mathcal{R}_i$ and a polyhedral cone $\mathcal{C}_i$. We claim that $\mathcal{E}_i \cap \mathcal{R}_i + \mathcal{C}_i = \mathcal{E}_i \cap \mathcal{P}_i$.

Let $x \in \mathcal{E}_i \cap \mathcal{R}_i + \mathcal{C}_i$. Note that $\mathcal{R}_i + \mathcal{C}_i \subseteq \mathcal{P}_i$ so that $x \in \mathcal{P}_i$ and since $\mathcal{C}_i \subseteq \mathrm{rec}(\mathcal{E}_i)$, we have $x \in \mathcal{E}_i$ as well. Thus, $\mathcal{E}_i \cap \mathcal{R}_i + \mathcal{C}_i \subseteq \mathcal{E}_i \cap \mathcal{P}_i$. Let $x \in \mathcal{E}_i \cap \mathcal{P}_i$. Then $x \in \mathcal{B}_i \cap \mathcal{P}_i = \mathcal{R}_i + \mathcal{C}_i$ and we may write $x = r + c$ for some $r \in \mathcal{R}_i$, $c \in \mathcal{C}_i$. Note that $c \in \mathrm{rec}(\mathcal{E}_i)$, and since $\mathrm{rec}(\mathcal{E}_i)$ is a linear space by Lemma 3, we obtain $-c \in \mathrm{rec}(\mathcal{E}_i)$ as well. Then $x = (x - c) + c$ and $x - c = r \in \mathcal{E}_i \cap \mathcal{R}_i$, $c \in \mathcal{C}_i$ so $x \in \mathcal{E}_i \cap \mathcal{R}_i + \mathcal{C}_i$. $\diamond$

Claim 3. *We can assume without loss of generality* $p = 1$.

Proof of claim. Let $\mathcal{E} \cap \mathcal{P} \subseteq \mathbb{R}^{n+p}$. We prove that $S = \mathrm{proj}_n(\mathcal{E} \cap \mathcal{P})$ has the desired decomposition (7), by induction on p. For this claim, we assume the base case, $p = 1$. Now let $p = m$, and suppose the statement holds for $p < m$.

Given $\mathcal{E} \cap \mathcal{P} \subseteq \mathbb{R}^{n+m}$, by the base case $p = 1$ there exist ellipsoidal regions $\mathcal{E}'_i$ and polyhedra $\mathcal{P}'_i$ such that

$$\operatorname{proj}_{n+m-1}(\mathcal{E} \cap \mathcal{P}) = \bigcup_{i=1}^{t} (\mathcal{E}'_i \cap \mathcal{P}'_i).$$

Since the projection of a union is the union of the projections, we have

$$S = \operatorname{proj}_n(\operatorname{proj}_{n+m-1}(\mathcal{E} \cap \mathcal{P})) = \bigcup_{i=1}^{t} \operatorname{proj}_n(\mathcal{E}'_i \cap \mathcal{P}'_i).$$

Then by the induction hypothesis there exists ellipsoidal regions $\mathcal{E}'_{i,j}$ and polyhedra $\mathcal{P}'_{i,j}$ such that

$$S = \bigcup_{i=1}^{t} \Big(\bigcup_{j=1}^{t_i} \mathcal{E}'_{i,j} \cap \mathcal{P}'_{i,j} \Big),$$

and we are done. ◇

To prove Proposition 1 it remains to show the following. Assume we are given $\mathcal{E} \cap \mathcal{P} \subseteq \mathbb{R}^{n+1}$. We must show the existence of ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, and polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$, for $i = 1, \dots, k$, that satisfy (7).

Given a half-space $H^+ = \{x \in \mathbb{R}^n \mid a^\top x \geq b\}$, we write H for the hyperplane $\{x \in \mathbb{R}^n \mid a^\top x = b\}$ and H^- for the half-space $\{x \in \mathbb{R}^n \mid a^\top x \leq b\}$. A polyhedron is the intersection of finitely many half-spaces. Thus, there exist half-spaces $H_1^+, \dots, H_s^+ \subseteq \mathbb{R}^{n+1}$ such that $\mathcal{P} = \cap_{i=1}^s H_i^+$.

We will define a collection $\mathcal{H}$ of hyperplanes that will allow us to compute $\operatorname{proj}_n(\mathcal{E} \cap \mathcal{P})$. By Lemma 4, there exists a hyperplane $H_0 \subset \mathbb{R}^{n+1}$ with $e_{n+1} \notin \operatorname{lin}(H_0)$ such that $\operatorname{proj}_n(\mathcal{E}) = \operatorname{proj}_n(\mathcal{E} \cap H_0)$. We arbitrarily pick one closed half-space defined by H_0 to be H_0^+ and the other to be H_0^-. Then

$$\mathcal{E} \cap \mathcal{P} = (\mathcal{E} \cap H_0^+ \cap_{i=1}^s H_i^+) \cup (\mathcal{E} \cap H_0^- \cap_{i=1}^s H_i^+),$$

and it suffices to show the existence of ellipsoidal regions and polyhedra satisfying (7) for one of the regions $\mathcal{E} \cap H_0^+ \cap_{i=1}^s H_i^+$ or $\mathcal{E} \cap H_0^- \cap_{i=1}^s H_i^+$. By symmetry, we show this existence for $\mathcal{E} \cap H_0^+ \cap_{i=1}^s H_i^+$. Let

$$\mathcal{H} := \Big\{ H_i \;\Big|\; i \in \{1, \dots, s\},\ e_{n+1} \notin \operatorname{lin}(H_i) \Big\} \cup \Big\{ H_0 \Big\}.$$

Claim 4. *We have*

$$\operatorname{proj}_n(\mathcal{E} \cap_{i=0}^s H_i^+) = \bigcup_{H \in \mathcal{H}} \operatorname{proj}_n(\mathcal{E} \cap H \cap_{i=0}^s H_i^+).$$

Proof of claim. The right hand side is clearly contained in the left hand side, so it suffices to show the forward containment. It suffices to show that $\mathcal{E} \cap_{i=0}^s H_i^+$ has the following property: for any $x \in \mathcal{E} \cap_{i=0}^s H_i^+$ there exists a hyperplane $H \in \mathcal{H}$ and a $\lambda \in \mathbb{R}$ such that $x + \lambda e_{n+1} \in \mathcal{E} \cap H \cap_{i=0}^s H_i^+$.

Let $\bar{x} \in \mathcal{E} \cap_{i=0}^{s} H_i^+$. To prove the claim, we show that we can translate $\bar{x}$ along $\pm e_{n+1}$, and inside the feasible region, until it meets a half-space in $\mathcal{H}$ at equality. By the existence of the shadowing hyperplane H_0, there is one direction among $\pm e_{n+1}$ along which $\bar{x}$ may be translated to intersect H_0 while staying inside $\mathcal{E}$. That is, there exists $\bar{\lambda} \in \mathbb{R}$ such that $\bar{x}+\bar{\lambda}e_{n+1} \in \mathcal{E} \cap H_0$. Then, there exists a possibly different $\lambda' \in \mathbb{R}$ with the same sign as $\bar{\lambda}$ and $|\lambda'| \leq |\bar{\lambda}|$ such that $\bar{x}+\lambda' e_{n+1} \in \mathcal{E} \cap_{i=0}^{s} H_i^+$ and $\bar{x}+\lambda' e_{n+1}$ lies on at least one hyperplane $H \in \mathcal{H}$. $\diamond$

Now it suffices to show that for any $H \in \mathcal{H}$ there exists an ellipsoidal region $\mathcal{E}' \subseteq \mathbb{R}^n$ and a polyhedron $\mathcal{P}' \subseteq \mathbb{R}^n$ such that

$$\text{proj}_n(\mathcal{E} \cap H \cap_{i=0}^{s} H_i^+) = \mathcal{E}' \cap \mathcal{P}'.$$

Without loss of generality, we may assume that $H_i \cap H \neq \emptyset$ for each $i = 0, \ldots, s$. If not, say $H_i \cap H = \emptyset$ for some $i \in \{0, \ldots, s\}$, i.e., the hyperplanes H_i and H are parallel. Then either $\mathcal{E} \cap H \cap H_i^+ = \emptyset$ and our region is empty, or $\mathcal{E} \cap H \cap H_i^+ = \mathcal{E} \cap H$ and H_i^+ is redundant and may be removed.

We now show that each half-space H_i^+ for $i \in \{0, \ldots, s\}$, with H_i different from H, can be replaced with a different half-space M_i^+ such that $\mathcal{E} \cap H \cap H_i^+ = \mathcal{E} \cap H \cap M_i^+$ and $e_{n+1} \in \text{lin}(M_i^+)$. Fix i such that $i \in \{0, \ldots, s\}$ and $H_i \neq H$. Let $M_i := H \cap H_i + \text{span}(e_{n+1})$. Since $e_{n+1} \notin \text{lin}(H)$, we have that M_i is a hyperplane in $\mathbb{R}^{n+1}$ that divides H into the same two regions that H_i does. In particular, upon choice of direction, we have that M_i^+ has the desired properties.

We are now ready to describe the polyhedron $\mathcal{P}'$. First, remove from the intersection $\mathcal{E} \cap H \cap_{i=0}^{s} H_i^+$ any redundant H_i^+ and the H_i^+ such that $H_i = H$. Then upon relabeling we may rewrite $\mathcal{E} \cap H \cap_{i=0}^{s} H_i^+$ as $\mathcal{E} \cap H \cap_{i=0}^{s'} H_i^+$. We may now replace each H_i^+ with M_i^+. By the requirement $e_{n+1} \in \text{lin}(M_i^+)$, we have that each M_i^+ is defined by a linear inequality with the coefficient of x_{n+1} equal to 0. Thus, the projection $\text{proj}_n(M_i^+)$ is a half-space in $\mathbb{R}^n$ which we denote $\bar{H}_i^+$. Further, if each H_i^+ for $i = 0, \ldots, s'$ is replaced in this way, we have

$$\text{proj}_n(\mathcal{E} \cap H \cap_{i=0}^{s'} H_i^+) = \text{proj}_n(\mathcal{E} \cap H \cap_{i=0}^{s'} M_i^+) = \text{proj}_n(\mathcal{E} \cap H) \cap_{i=0}^{s'} \bar{H}_i^+,$$

and we have the desired polyhedron $\mathcal{P}' := \cap_{i=0}^{s'} \bar{H}_i^+$.

It remains to show that $\text{proj}_n(\mathcal{E} \cap H)$ is an ellipsoidal region $\mathcal{E}' \subseteq \mathbb{R}^n$. Let $H = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid a^\top(x, y) = b\}$. Consider the linear transformation from $\mathbb{R}^{n+1}$ to itself, defined by the matrix A whose first n rows are the first n standard unit vectors of $\mathbb{R}^{n+1}$ and whose last row is a. Note that A is invertible since e_{n+1} is not in $\text{lin}(H)$, i.e., $a_{n+1} \neq 0$. Then, by the definition of A, for any vector $(x, y) \in \mathbb{R}^{n+1}$ we have $A(x, y) = (x, c)$ where $c = a^\top(x, y)$. It follows that $A(H)$ gets mapped to the hyperplane $\{(x, y) \in \mathbb{R}^{n+1} \mid y = b\}$. Now, since A is invertible we have

$$\begin{aligned} x \in \text{proj}_n(\mathcal{E} \cap H) &\Leftrightarrow \exists y \in \mathbb{R} \text{ such that } (x, y) \in \mathcal{E} \cap H \\ &\Leftrightarrow (x, b) \in A(\mathcal{E} \cap H) \\ &\Leftrightarrow (x, b) \in A(\mathcal{E}). \end{aligned}$$

This shows that $\mathrm{proj}_n(\mathcal{E} \cap H) = A(\mathcal{E})|_{y=b}$. Ellipsoidal regions are clearly preserved under invertible linear transformations, therefore $A(\mathcal{E})$ is an ellipsoidal region. Finally, by Lemma 1, the set $A(\mathcal{E})|_{y=b}$ is an ellipsoidal region. This concludes the proof that $\mathrm{proj}_n(\mathcal{E} \cap H)$ is an ellipsoidal region $\mathcal{E}'$. □ □

We remark that all of the statements in this section behave nicely with respect to rationality. In greater detail, if the given ellipsoidal regions, polyhedra, and vectors are rational, then the resulting objects are also all rational. This observation can be seen directly from the proofs of these results. In particular, the rational version of Proposition 1 has the following statement.

Proposition 2. *Let $\mathcal{E} \subseteq \mathbb{R}^{n+p}$ be a nonempty rational ellipsoidal region and $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ be a nonempty rational polyhedron. Let $S = \mathrm{proj}_n(\mathcal{E} \cap \mathcal{P})$. Then there exist rational ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, rational polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and a rational polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that*

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C},$$

where $\mathcal{C} = \mathrm{proj}_n(\mathrm{rec}(\mathcal{E}) \cap \mathrm{rec}(\mathcal{P}))$.

3 Proofs of main results

We begin this section with a proposition that establishes the sufficiency of the conditions given in Theorem 1.

Proposition 3. *Let $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$ be ellipsoidal regions, $\mathcal{P}_i \subseteq \mathbb{R}^n, i = 1, \ldots k$, be polytopes and $\mathcal{C} \subseteq \mathbb{R}^n$ a polyhedral cone. Suppose*

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}.$$

Then S is binary EMI-representable.

Proof. Assume that we are given a set

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C},$$

where $\mathcal{E}_i = \{x \in \mathbb{R}^n \mid (x - c_i)^\top Q_i (x - c_i) \leq \gamma_i\}$ are ellipsoidal regions, $\mathcal{P}_i = \{x \in \mathbb{R}^n \mid A_i x \leq b_i\}$ are polytopes, and $\mathcal{C} = \mathrm{cone}\{r^1, \ldots, r^t\} \subseteq \mathbb{R}^n$ is a polyhedral cone. For each ellipsoidal region $\mathcal{E}_i$, if $\gamma_i > 0$ we can normalize the right hand side of the inequality to 1. Else, either $\gamma_i < 0$ and $\mathcal{E}_i$ is empty or $\gamma_i = 0$ and $\mathcal{E}_i$ is an affine space. In the case, that $\mathcal{E}_i$ is an affine space, we may set $\gamma_i = 1$ and add the linear equalities defining $\mathcal{E}_i$ to the system $A_i x \leq b_i$ defining $\mathcal{P}_i$. Thus, we may assume $\gamma_i = 1$ for all $i = 1, \ldots, k$.

We introduce new continuous variables $x^i \in \mathbb{R}^n$ and binary variables $\delta_i \in \{0,1\}$, for $i = 1,\ldots,k$, that will model the individual regions $\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}$. Then S can be described as the set of $x \in \mathbb{R}^n$ such that

$$
\begin{aligned}
& x = \sum_{i=1}^{k} (x^i + \delta_i c_i) + \sum_{j=1}^{t} \lambda_j r^j && \\
& A_i x^i \leq \delta_i (b_i - A_i c_i) && i = 1,\ldots,k \\
& \sum_{i=1}^{k} \delta_i = 1 && \\
& \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix}^{\top} \begin{pmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_k \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix} \leq 1 && \\
& x^i \in \mathbb{R}^n, \ \delta_i \in \{0,1\} && i = 1,\ldots,k \\
& \lambda_j \in \mathbb{R}_{\geq 0} && j = 1,\ldots,t.
\end{aligned}
$$

Now if $\delta_1 = 1$ the remaining δ_i must be 0. Then for each x^i with $i \neq 1$, we have the constraint $A_i x^i \leq 0$ which has the single feasible point $x^i = 0$ since $\mathcal{P}_i$ is a polytope. The remaining constraints reduce to

$$
\begin{aligned}
& x = x^1 + c_1 + \sum_{j=1}^{t} \lambda_j r^j && \\
& A_1(x^1 + c_1) \leq b_1 && \\
& (x^1)^{\top} Q_1 x^1 \leq 1 && \\
& x^1 \in \mathbb{R}^n && \\
& \lambda_j \in \mathbb{R}_{\geq 0} && j = 1,\ldots,t.
\end{aligned}
$$

By employing a change of variables $x' = x^1 + c_1$, it can be checked that the latter system describes the region $\mathcal{E}_1 \cap \mathcal{P}_1 + \mathcal{C}$. The remaining regions follow symmetrically. Therefore S is binary EMI-representable. □ □

A similar proposition holds that proves sufficiency of the conditions given in Theorem 2.

Proposition 4. *Let $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1,\ldots,k$ be rational ellipsoidal regions, $\mathcal{P}_i \subseteq \mathbb{R}^n, i = 1,\ldots k$, be rational polytopes and $r^i \in \mathbb{Z}^n$, $i = 1,\ldots,t$ be integral vectors. Suppose*

$$
S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \text{int.cone}\{r^1,\ldots,r^t\}.
$$

Then S is rational EMI-representable.

The proof of Proposition 4 is identical to the proof of Proposition 3 except that the constraints $\lambda_j \in \mathbb{R}_{\geq 0}$ are replaced with $\lambda_j \in \mathbb{Z}_{\geq 0}$ and the binary constraints $\delta_i \in \{0, 1\}$ are replaced with $0 \leq \delta_i \leq 1$ and $\delta_i \in \mathbb{Z}$.

We are now ready to prove the two main theorems.

Proof of Theorem 1 Sufficiency of the conditions follows by Proposition 3. The remainder of the proof is devoted to proving necessity of the condition. We are given an ellipsoidal region $\mathcal{E}$ and a polyhedron $\mathcal{P}$ in $\mathbb{R}^{n+p+q}$, and we define

$$\begin{aligned} \bar{S} &:= \mathcal{E} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0,1\}^q), \\ S &:= \mathrm{proj}_n(\bar{S}). \end{aligned}$$

We may assume $\bar{S} \neq \emptyset$, else the statement follows immediately. We must show the existence of ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}.$$

Let T be the subset of $\{0,1\}^q$ such that $\bar{z} \in T$ if and only if $\bar{S}|_{z=\bar{z}} \neq \emptyset$. It follows that $S = \bigcup_{\bar{z} \in T} \mathrm{proj}_n(\bar{S}|_{z=\bar{z}})$. Let $\bar{z} \in T$. Then $\bar{S}|_{z=\bar{z}} = \mathcal{E}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}}$. It is well known that $\mathrm{rec}(\mathcal{P}|_{z=\bar{z}})$ is independent of choice of $\bar{z} \in T$. Moreover, it follows from (5) in Lemma 1 and from Lemma 2 that $\mathrm{rec}(\mathcal{E}|_{z=\bar{z}})$ is also independent of choice of $\bar{z} \in T$.

We can now apply Proposition 1 and observe that

$$S = \bigcup_{\bar{z} \in T} \Big(\bigcup_{i \in I_{\bar{z}}} \mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C} \Big),$$

where $\mathcal{C}$ is a polyhedral cone independent of the choice of $\bar{z} \in T$. Now for any sets $A_1, \ldots, A_t, B \subseteq \mathbb{R}^n$, it can be checked that the Minkowski sum satisfies the relation

$$\bigcup_{i=1}^{t} (A_i + B) = (\bigcup_{i=1}^{t} A_i) + B.$$

It follows that

$$S = \bigcup_{\bar{z} \in T} \bigcup_{i \in I_{\bar{z}}} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}.$$

Since each of the sets $I_{\bar{z}}$ are finite, this completes the proof of Theorem 1. □

Proof of Theorem 2 Sufficiency of the conditions follows by Proposition 4. The remainder of the proof is devoted to proving necessity of the condition. We are given a rational ellipsoidal region $\mathcal{E}$ and a rational polyhedron $\mathcal{P}$ in $\mathbb{R}^{n+p+q}$, and we define

$$\begin{aligned} \bar{S} &:= \mathcal{E} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q), \\ S &:= \mathrm{proj}_n(\bar{S}). \end{aligned}$$

We may assume $\bar{S} \neq \emptyset$, else the statement follows immediately. We must show the existence of rational ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, rational polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, k$, and integral vectors $r^1, \ldots, r^t \in \mathbb{Z}^n$ such that

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \text{int.cone}\{r^1, \ldots, r^t\}.$$

We first show that we can decompose $\bar{S}$ into a bounded region and an integer cone.

Claim 1. *There exists a rational polytope $\mathcal{R} \subseteq \mathbb{R}^{n+p+q}$ and integral vectors $r^1, \ldots, r^t \in \mathbb{Z}^{n+p+q}$ such that*

$$\bar{S} = \mathcal{E} \cap \mathcal{R} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q) + \text{int.cone}\{r^1, \ldots, r^t\}.$$

Proof of claim. Let $\mathcal{B} \subseteq \mathbb{R}^{n+p+q}$ be a rational polyhedral approximation of $\mathcal{E}$ as in Observation 2 such that $\mathcal{B}$ is a rational polyhedron, $\mathcal{E} \subseteq \mathcal{B}$, and $\text{rec}(\mathcal{E}) = \text{rec}(\mathcal{B})$. Then $\mathcal{E} \cap \mathcal{P} = \mathcal{E} \cap (\mathcal{B} \cap \mathcal{P})$. Since $\mathcal{B} \cap \mathcal{P}$ is a rational polyhedron, we can decompose $\mathcal{B} \cap \mathcal{P} = \mathcal{R}' + \mathcal{C}$ for some rational polytope $\mathcal{R}'$ and a rational polyhedral cone $\mathcal{C}$. Since $\mathcal{C}$ is rational, there exist integral vectors $r^1, \ldots, r^t \in \mathbb{Z}^{n+p+q}$ such that $\mathcal{C} = \text{cone}\{r^1, \ldots, r^t\}$. Note that each $r^i \in \text{rec}(\mathcal{E})$. Let

$$\mathcal{R} := \mathcal{R}' + \Big\{ \sum_{i=1}^{t} \lambda_i r^i \;\Big|\; 0 \leq \lambda_i \leq 1 \text{ for each } i = 1, \ldots, t \Big\}.$$

It is well-known that $\mathcal{B} \cap \mathcal{P} = \mathcal{R}' + \mathcal{C} = \mathcal{R} + \text{int.cone}\{r^1, \ldots, r^t\}$, see for example the proof of Theorem 4.30 in [1].

We now show that $\mathcal{R}$ meets the conditions of the claim. Let $p \in \bar{S}$. Then $p \in \mathcal{B} \cap \mathcal{P}$ so $p = q + \sum_{i=1}^{t} \mu_i r^i$ for some $q \in \mathcal{R}$ and $\mu_i \in \mathbb{Z}_{\geq 0}$. Since $p \in \mathbb{R}^{n+p} \times \mathbb{Z}^q$ and μ_i, r^i are integral, we have $q \in \mathbb{R}^{n+p} \times \mathbb{Z}^q$. Moreover, $q \in \mathcal{E}$ since $p \in \mathcal{E}$ and $\text{rec}(\mathcal{E})$ is a linear space and each $r^i \in \text{rec}(\mathcal{E})$. Thus, $\bar{S} \subseteq \mathcal{E} \cap \mathcal{R} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q) + \text{int.cone}\{r^1, \ldots, r^t\}$.

For the reverse inclusion, let $q \in \mathcal{E} \cap \mathcal{R} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q)$ and $\mu_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, t$. Let $p = q + \sum_{i=1}^{t} \mu_i r^i$. Since $q \in \mathbb{R}^{n+p} \times \mathbb{Z}^q$ and μ_i, r^i are integral, we have $p \in \mathbb{R}^{n+p} \times \mathbb{Z}^q$. Also, each $r^i \in \text{rec}(\mathcal{E})$ which implies that $p \in \mathcal{E}$. Finally, $p \in \mathcal{R} + \mathcal{C} = \mathcal{B} \cap \mathcal{P} \subseteq \mathcal{P}$ which implies $p \in \mathcal{P}$. Therefore, $p \in \bar{S}$. $\diamond$

Let $\bar{r}^1, \ldots, \bar{r}^t \in \mathbb{Z}^n$ be the vectors consisting of the first n components of $r^1, \ldots, r^t$. Then by linearity of the projection operator, we have

$$S = \text{proj}_n(\mathcal{E} \cap \mathcal{R} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q)) + \text{int.cone}\{\bar{r}^1, \ldots, \bar{r}^t\}.$$

Let $\bar{S}' := \mathcal{E} \cap \mathcal{R} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q)$. Let T be the subset of $\mathbb{Z}^q$ such that $\bar{z} \in T$ if and only if $\bar{S}'|_{z=\bar{z}} \neq \emptyset$ and note that T is finite, since $\mathcal{R}$ is a polytope. It follows that

$$S = \bigcup_{\bar{z} \in T} \text{proj}_n(\bar{S}'|_{z=\bar{z}}) + \text{int.cone}\{\bar{r}^1, \ldots, \bar{r}^t\}.$$

Let $\bar{z} \in T$. Then $\bar{S}'|_{z=\bar{z}} = \mathcal{E}|_{z=\bar{z}} \cap \mathcal{R}|_{z=\bar{z}}$. We can now apply Proposition 2 and observe that

$$S = \bigcup_{\bar{z} \in T} \bigcup_{i \in I_{\bar{z}}} (\mathcal{E}_i \cap \mathcal{P}_i) + \text{int.cone}\{\bar{r}^1, \ldots, \bar{r}^t\}.$$

Note that since each $\mathcal{R}|_{z=\bar{z}}$ is a polytope, there is no cone $\mathcal{C}$. Since each of the sets $I_{\bar{z}}$ are finite, this completes the proof of Theorem 2 □

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