On the Mixed Binary Representability of Ellipsoidal Regions

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Abstract. Representability results for mixed-integer linear systems play a fundamental role in optimization since they give geometric characterizations of the feasible sets that arise from mixed-integer linear programs. We consider a natural extension of mixed-integer linear systems obtained by adding just one ellipsoidal inequality. The set of points that can be described, possibly using additional variables, by these systems are called ellipsoidal mixed binary representable. In this work, we give geometric conditions that characterize ellipsoidal mixed binary representable sets.

1 Introduction

The theory of representability starts with a paper of Dantzig [1] and studies one fundamental question: Given a specified type of algebraic constraints, which subsets of $\mathbb{R}^n$ can be represented as the feasible points of a system defined by these constraints, possibly using additional variables? Several researchers have investigated representability questions (see, e.g., [3], [4], [8], [9], [5], [10], [7], [6]), and a systematic study for mixed-integer linear systems is mainly due to Meyer and Jeroslow.

Since the projection of a polyhedron is a polyhedron (see [11]), the sets representable by system of linear inequalities are polyhedra. More formally, a set $S \subseteq \mathbb{R}^n$ is representable as the projected solution set of a linear system

$$\begin{aligned} &Dw \leq d \\ &w \in \mathbb{R}^n \times \mathbb{R}^p \end{aligned}$$

if and only if S is a polyhedron.

If we allow also binary extended variables, a geometric characterization has been given by Jeroslow [6]. A set $S \subseteq \mathbb{R}^n$ is representable as the projected solution set of a mixed-integer linear system

$$\begin{aligned} &Dw \leq d \\ &w \in \mathbb{R}^n \times \mathbb{R}^p \times \{0,1\}^p \end{aligned}$$

if and only if S is the union of a finite number of polyhedra, each having the same recession cone.

We are interested in giving representability results for mixed-integer sets defined not only by linear inequalities, but also by quadratic inequalities of the form $(w-c)^\top Q(w-c) \le \gamma$, where Q is a positive semidefinite matrix. Inequalities of this type are called *ellipsoidal inequalities*, and the set of points that satisfy one of them is called an *ellipsoidal region.* Ellipsoidal inequalities arise in many practical applications. As an example, many real-life quantities are normally distributed; and for a normal distribution, a natural confidence set, containing the vast majority of the objects, is an ellipsoidal region. See, e.g., [12] for other applications of ellipsoidal inequalities.

A characterization of sets representable by an arbitrary number of ellipsoidal inequalities seems to be currently completely out of reach. In fact, it is easy to construct examples where just two ellipsoidal inequalities in $\mathbb{R}^3$ project to a semialgebraic set described by polynomials of degree four in $\mathbb{R}^2$. This can happen even without linear inequalities or binary extended variables. As a consequence, in this work we will focus on understanding the expressive power of just one ellipsoidal inequality.

Formally, we say that a set $S \subseteq \mathbb{R}^n$ is *ellipsoidal mixed binary (EMB) representable* if it can be obtained as the projection onto $\mathbb{R}^n$ of the solution set of a system of the form

$$\begin{aligned} &Dw \le d \\ &(w-c)^\top Q(w-c) \le \gamma \\ &w \in \mathbb{R}^{n+p} \times \{0,1\}^q, \end{aligned} \tag{1}$$

where Q is positive semidefinite. There is a strong connection between EMB-representable sets and mixed-integer quadratic programming (MIQP). In a MIQP problem we aim at minimizing a quadratic function over mixed integer points in a polyhedron. Since MIQP $\in \mathcal{NP}$ [2], any MIQP with bounded objective is polynomially equivalent to a polynomial number of MIQP feasibility problems. If the objective quadratic is ellipsoidal, then each feasibility problem is a feasibility problem over a set of the form (1).

Our main result is the following geometric characterization of EMB-representable sets.

Theorem 1. *A set $S \subseteq \mathbb{R}^n$ is EMB-representable if and only if there exist ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that*

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}. \tag{2}$$

An example of an EMB-representable set is given in Figure 1.

Both directions of Theorem 1 have geometric implications. Since each set (2) can be obtained as the projection of a set described by a system (1), this

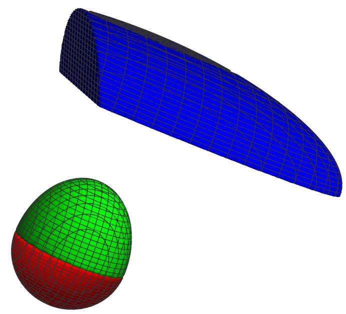

Fig. 1. An EMB-representable set in $\mathbb{R}^3$

means that the k ellipsoidal regions $\mathcal{E}_i$ can be expressed with just one ellipsoidal inequality in a higher dimension. We prove this direction of the theorem by explicitly giving an extended formulation for S.

The other direction of Theorem 1 states that the projection of each system (1) onto $\mathbb{R}^n$ is a set of the form (2). The proof of this statement essentially reduces to proving that the projection S of a set $\{x \in \mathbb{R}^{n+1} \mid Dx \leq d,\ (x-c)^\top Q(x-c) \leq \gamma\}$ onto $\mathbb{R}^n$ is a set of the form (2). In order to do so, we introduce the key concept of a *shadowing hyperplane*. This hyperplane, that will be formally introduced later, allows us to split the ellipsoidal region into two 'parts' which, in turn, allow us to decompose S as a union of subsets S_i. We will then see how each set S_i can be obtained as the projection of a set in $\mathbb{R}^{n+1}$ lying on a hyperplane. This will allow us to prove that each S_i can be described with linear inequalities and one ellipsoidal inequality.

The remainder of this paper is organized as follows. In §2, we provide a number of results relating to the intersection of an ellipsoidal region with a polyhedron and the projections of such regions. In §3, we prove Theorem 1.

2 Ellipsoidal Regions and Hyperplanes

In this section we formally define ellipsoidal regions. These regions will appear throughout our study of representability. We will prove a few results on the intersection of ellipsoidal regions with half-spaces as well as their projections. These results will be necessary for our proof of Theorem 1.

We say that a set $\mathcal{E}$ is an *ellipsoidal region* in $\mathbb{R}^n$ if there exists an $n \times n$ matrix $Q \succeq 0$ (i.e., Q is positive semi-definite), a vector $c \in \mathbb{R}^n$, and a number $\gamma \in \mathbb{R}$, such that

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid (x-c)^\top Q(x-c) \leq \gamma\}.$$

We note that if $Q \succ 0$ (i.e., Q is positive definite) and $\gamma > 0$, then $\mathcal{E}$ is an *ellipsoid*, i.e., the image of the unit ball $\mathcal{B} = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ under an invertible affine transformation.

Given a set $E \subseteq \mathbb{R}^n \times \mathbb{R}^p$ and a vector $\bar{y} \in \mathbb{R}^p$, we define the *$\bar{y}$-restriction* of E as

$$E|_{y=\bar{y}} = \{x \in \mathbb{R}^n \mid (x, \bar{y}) \in E\}.$$

Note that $E|_{y=\bar{y}}$ geometrically consists of the intersection of E with coordinate hyperplanes. Sometimes we will need to consider $E|_{y=\bar{y}}$ in the original space $\mathbb{R}^n \times \mathbb{R}^p$, thus we also define

$$\tilde{E}|_{y=\bar{y}} = \{(x, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^p \mid (x, \bar{y}) \in E\}.$$

We will also need to perform several restrictions $y_1 = \bar{y}_1, \dots, y_k = \bar{y}_k$ at the same time. In such case we simply write $E|_{y_1=\bar{y}_1,\dots,y_k=\bar{y}_k}$ and $\tilde{E}|_{y_1=\bar{y}_1,\dots,y_k=\bar{y}_k}$.

In the remainder of the paper we will denote by "rec" the recession cone of a set, by "lin" the lineality space of a set, by "span" the linear space generated by a set of vectors, by "cone" the cone generated by a set of vectors, by "range" the range of a matrix, and by "ker" the kernel of a matrix.

The following observation is well-known, and we give a proof for completeness.

Observation 1 *Let $q(x) = x^\top Q x + b^\top x$ be a quadratic function on $\mathbb{R}^n$ with $Q \succeq 0$. Then $q(x)$ has a minimum on $\mathbb{R}^n$ if and only if b is in the range of Q.*

Proof. Assume $b \notin \text{range}(Q)$. Then since Q is symmetric, we can write $b = Qr + c$ with $Qc = 0$ and $c \neq 0$. Consider $x(t) = -tc$ for $t \in \mathbb{R}$. Then we have

$$q(x(t)) = b^\top x(t) = -tc^\top c.$$

Since $c \neq 0$, we see that $q(x(t)) \to -\infty$ as $t \to +\infty$. Thus, $q(x)$ has no minimum on $\mathbb{R}^n$.

Assume there exists $x_0 \in \mathbb{R}^n$ such that $\frac{1}{2}b = Qx_0$. Then

$$q(x) = (x + x_0)^\top Q(x + x_0) - x_0^\top Q x_0$$

and $q(x)$ has a minimum at any $\bar{x}$ such that $\bar{x} + x_0 \in \ker(Q)$. In particular, $-x_0$ is a minimizer and $q(-x_0) = -x_0^\top Q x_0$ is the optimal value. □

The following lemma shows that ellipsoidal regions are closed under intersections with coordinate hyperplanes. This is equivalent to fixing a number of variables.

Lemma 1. *Let $\mathcal{E}$ be an ellipsoidal region in $\mathbb{R}^n \times \mathbb{R}^p$. Then for any $\bar{y} \in \mathbb{R}^p$, the set $\mathcal{E}|_{y=\bar{y}}$ is an ellipsoidal region in $\mathbb{R}^n$.*

Proof. Let $\mathcal{E} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid q(x, y) \leq \gamma\}$, where $q(x, y)$ is the quadratic polynomial

$$q(x, y) = \begin{pmatrix} x - c \\ y - c' \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & \bar{Q} \end{pmatrix} \begin{pmatrix} x - c \\ y - c' \end{pmatrix}.$$

For any fixed $\bar{y} \in \mathbb{R}^p$, since $Q \succeq 0$ it suffices to show there exists $c_{\bar{y}} \in \mathbb{R}^n$ and $\gamma_{\bar{y}} \in \mathbb{R}$ such that

$$\mathcal{E}|_{y=\bar{y}} = \{x \in \mathbb{R}^n \mid (x - c_{\bar{y}})^\top Q(x - c_{\bar{y}}) \leq \gamma_{\bar{y}}\}.$$

Let $\bar{y} \in \mathbb{R}^p$. Since $q(x, y)$ has a minimum on $\mathbb{R}^n \times \mathbb{R}^p$, the quadratic function

$$q(x, \bar{y}) = (x - c)^\top Q(x - c) + 2(\bar{y} - c')^\top R^\top (x - c) + (\bar{y} - c')^\top \bar{Q}(\bar{y} - c'),$$

has a minimum on $\mathbb{R}^n$. Applying Observation 1, $R(\bar{y} - c') \in \text{range}(Q)$, and there exists $\bar{x} \in \mathbb{R}^n$ such that $Q\bar{x} = R(\bar{y} - c')$. Defining $c_{\bar{y}} := c - \bar{x}$ and $\gamma_{\bar{y}} := \gamma + \bar{x}^\top Q\bar{x} - (\bar{y} - c')^\top \bar{Q}(\bar{y} - c')$ we have

$$\mathcal{E}|_{y=\bar{y}} = \{x \in \mathbb{R}^n \mid (x - c_{\bar{y}})^\top Q(x - c_{\bar{y}}) \leq \gamma_{\bar{y}}\}.$$

□

We are now ready to provide a geometric description of ellipsoidal regions. A consequence of this description is that any non-empty ellipsoidal region may be decomposed into the Minkowski sum of an ellipsoid and a linear space.

Lemma 2. *Let $\mathcal{E}$ be an ellipsoidal region in $\mathbb{R}^n$. Then*

(i) $\mathcal{E} = \emptyset$, *or*
(ii) $\mathcal{E}$ *is an affine space, or*
(iii) There exists a k-dimensional linear space $L \subseteq \mathbb{R}^n$, and k distinct indices $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that the restriction

$$\mathcal{E}|_{x_{i_1}=\bar{x}_{i_1},\ldots,x_{i_k}=\bar{x}_{i_k}}$$

is an ellipsoid in $\mathbb{R}^{n-k}$, and

$$\mathcal{E} = \tilde{\mathcal{E}}|_{x_{i_1}=\bar{x}_{i_1},\ldots,x_{i_k}=\bar{x}_{i_k}} + L.$$

Proof. Let $\mathcal{E} = \{x \in \mathbb{R}^n \mid (x - c)^\top Q(x - c) \leq \gamma\}$. If $\gamma < 0$, then $\mathcal{E} = \emptyset$ since Q is positive semidefinite. Thus, we may assume that $\gamma \geq 0$ and $\mathcal{E}$ is non-empty.

We now show that $\text{rec}(\mathcal{E}) = \{x \in \mathbb{R}^n \mid x^\top Qx \leq 0\} = \ker(Q)$. Since $\mathcal{E}$ is a closed convex set, $\text{rec}(\mathcal{E})$ is equal to the set of recession directions at any point $x \in \mathcal{E}$. Consider the point $c \in \mathcal{E}$. Then for any $r \in \ker(Q)$ and $\lambda > 0$ we have $c + \lambda r \in \mathcal{E}$ since $\lambda^2 r^\top Qr = 0 \leq \gamma$. Assume $r \in \mathbb{R}^n$ is a recession direction from $c \in \mathcal{E}$. Let $Q = L^\top L$ be a Cholesky decomposition of Q. Then for any $\lambda > 0$ we have $\lambda^2 r^\top Qr = \lambda^2 ||Lr||^2 \leq \gamma$, which implies $Lr = 0$ and $r \in \ker(Q)$.

Now assume $\gamma = 0$. By the above argument, $x \in \mathcal{E}$ if and only if $x \in \{c\} + \ker(Q)$. Thus $\mathcal{E} = \{c\} + \ker(Q)$ is an affine space.

Assume now $\gamma > 0$. If Q is invertible then $\mathcal{E}$ is an ellipsoid and we are done. Thus, we may assume $L := \ker(Q)$ is nontrivial. Let $\mathcal{L} = \{l_1, \ldots, l_k\}$ be a basis for L. Extend $\mathcal{L}$ to a basis $\mathcal{L}'$ of $\mathbb{R}^n$ by adding a subset of the standard basis vectors $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. Let $\mathcal{J} \subseteq \{1, \ldots, n\}$ be the set of indices j for which

$e_j \in \mathcal{L}'$. Let $\{i_1, \ldots, i_k\} = \{1, \ldots, n\} - \mathcal{J}$. Consider $\mathcal{E}' := \mathcal{E}|_{x_{i_1}=0,\ldots,x_{i_k}=0}$ and $\tilde{\mathcal{E}}' := \tilde{\mathcal{E}}|_{x_{i_1}=0,\ldots,x_{i_k}=0}$.

We now show $\mathcal{E} = \tilde{\mathcal{E}}' + L$. Since $\tilde{\mathcal{E}}' \subseteq \mathcal{E}$ and $\operatorname{rec}(\mathcal{E}) = L$, we clearly have $\tilde{\mathcal{E}}' + L \subseteq \mathcal{E}$. Let $v \in \mathcal{E}$. Expanding v in the basis $\mathcal{L}'$, we have for some $l \in L$ and scalars $\alpha_j \in \mathbb{R}$, that $v = l + \sum_{j \in \mathcal{J}} \alpha_j e_j$. Since $L = \operatorname{rec}(\mathcal{E})$ we have $v - l = \sum_{j \in \mathcal{J}} \alpha_j e_j \in \mathcal{E}'$ and $\mathcal{E} \subseteq \tilde{\mathcal{E}}' + L$.

By Lemma 1, $\mathcal{E}'$ is an ellipsoidal region in $\mathbb{R}^{n-k}$. To show $\mathcal{E}'$ is an ellipsoid in $\mathbb{R}^{n-k}$ it remains to show that $\mathcal{E}'$ is full-dimensional and bounded. If $\mathcal{E}'$ is unbounded, then $\mathcal{E}'$ has some recession direction outside of L which contradicts the fact that $\operatorname{rec}(\mathcal{E}) = L$. We finally show that $\mathcal{E}'$ is full-dimensional. We first show that $\mathcal{E}$ is full-dimensional in $\mathbb{R}^n$. This follows since $\gamma > 0$ and there exists a vector, namely $c \in \mathbb{R}^n$, for which the continuous function $(x-c)^\top Q(x-c)$ has value 0. This implies that there exists an ϵ-ball around c, say $\mathcal{B}$, such that $\mathcal{B} \subseteq \mathcal{E}$. The fact that $\mathcal{E}'$ is full-dimensional follows by considering the intersection of $\mathcal{B} + L$ with $\mathcal{E}'$. □

We make the following remark about the proof of *(iii)* that will be used later. If one of the standard basis vectors of $\mathbb{R}^n$, say e_n, is not contained in L, then we may assume that x_n does not occur among the fixed variables $x_{i_1}, \ldots, x_{i_k}$. To see this, note that in completing the basis $\mathcal{L}$ of L to a basis of $\mathbb{R}^n$ we may first add the standard basis vector e_n to the set $\mathcal{L}$.

It can be shown that Lemma 2 is in fact an if and only if statement. It then provides a complete geometric characterization of ellipsoidal regions. The next observation gives a description of the recession cones that will be encountered in this paper.

Observation 2 *Let $\mathcal{P}$ be a polyhedron and $\mathcal{E}$ an ellipsoidal region in $\mathbb{R}^n$. Then $\operatorname{rec}(\mathcal{E} \cap \mathcal{P})$ is a polyhedral cone.*

Proof. Clearly, $\operatorname{rec}(\mathcal{E} \cap \mathcal{P}) = \operatorname{rec}(\mathcal{E}) \cap \operatorname{rec}(\mathcal{P})$. The set $\operatorname{rec}(\mathcal{P})$ is a polyhedral cone (see, e.g., [11]), and $\operatorname{rec}(\mathcal{E})$ is a linear space by Lemma 2. As a consequence $\operatorname{rec}(\mathcal{E} \cap \mathcal{P})$ is a polyhedral cone. □

The following lemma shows that to compute the projection of an ellipsoidal region $\mathcal{E}$ in $\mathbb{R}^n$, it suffices to consider the projection of $\mathcal{E} \cap H$ for a specific hyperplane $H \subseteq \mathbb{R}^n$. We will refer to such a hyperplane H as a *shadowing hyperplane*, as it contains enough information to completely describe the projection, or 'shadow', of $\mathcal{E}$.

Given a set $S \subseteq \mathbb{R}^n$, and a positive integer $k \leq n$, we will denote by $\operatorname{proj}_k(S)$ the projection of S onto its first k coordinates. Formally,

$$\operatorname{proj}_k(S) = \{x \in \mathbb{R}^k \mid \exists y \in \mathbb{R}^{n-k} \text{ with } (x, y) \in S\}.$$

Lemma 3. *Let $\mathcal{E}$ be an ellipsoidal region in $\mathbb{R}^n$. Then there exists a hyperplane $H \subseteq \mathbb{R}^n$ with $e_n \notin \operatorname{lin}(H)$ such that*

$$\operatorname{proj}_{n-1}(\mathcal{E}) = \operatorname{proj}_{n-1}(\mathcal{E} \cap H).$$

Proof. We first note that it suffices to find a hyperplane H such that for any $x \in \mathcal{E}$ there exists $\lambda \in \mathbb{R}$ such that $x + \lambda e_n \in \mathcal{E} \cap H$. The cases $\mathcal{E} = \emptyset$ and $\mathcal{E}$ an affine space are trivial. If $\mathcal{E} = \emptyset$ then any hyperplane H with $e_n \notin \text{lin}(H)$ satisfies the condition of the lemma. If $\mathcal{E} = v + L$ is an affine space, either $e_n \in \text{lin}(L)$ or $e_n \notin \text{lin}(L)$. If $e_n \in \text{lin}(L)$, we may take $H = \{x \in \mathbb{R}^n \mid x_n = 0\}$ since for any $\bar{x} \in \mathcal{E}$ there exists $\bar{\lambda} \in \mathbb{R}$, namely $\bar{\lambda} = -\bar{x}_n$, such that $\bar{x} + \bar{\lambda} e_n \in \mathcal{E} \cap H$. If $e_n \notin \text{lin}(H)$, then we may take H to be any hyperplane containing $\mathcal{E}$ with $e_n \notin \text{lin}(H)$.

We now show the lemma when $\mathcal{E}$ is an ellipsoid, say $\mathcal{E} = \{Ax + c \mid ||x|| \leq 1\}$ with A an invertible $n \times n$ matrix. Let $H' = \{x \in \mathbb{R}^n \mid x_n = 0\}$. Clearly, for any x in the standard unit ball $\mathcal{B}$ there exists $\lambda \in \mathbb{R}$ such that $x + \lambda e_n \in \mathcal{B} \cap H'$. Let U be an orthogonal transformation that maps the standard unit vector e_n to $\frac{A^{-1} e_n}{||A^{-1} e_n||}$. Let T be the invertible affine transformation defined by $T(x) = AU(x) + c$. We claim that $H := T(H')$ is an appropriate hyperplane. Let $\bar{x} \in \mathcal{E}$. Since $\mathcal{E} = A\mathcal{B} + c = T(\mathcal{B})$, we have $T^{-1}(\bar{x}) \in \mathcal{B}$. Then there exists $\bar{\lambda} \in \mathbb{R}$ such that $T^{-1}(\bar{x}) + \bar{\lambda} e_n \in \mathcal{B} \cap H'$. Applying T we have $\bar{x} + \frac{\bar{\lambda}}{||A^{-1} e_n||} e_n \in \mathcal{E} \cap H$. We have $e_n \notin \text{lin}(H)$, since otherwise $\text{proj}_{n-1}(\mathcal{E}) = \text{proj}_{n-1}(\mathcal{E} \cap H) \subseteq \text{proj}_{n-1}(H)$ would have dimension at most $n-2$, contradicting the full-dimensionality of $\mathcal{E}$.

Assume now that $\mathcal{E}$ is a full-dimensional and unbounded ellipsoidal region, say $\mathcal{E} = \{x \in \mathbb{R}^n \mid (x-c)^\top Q (x-c) \leq \gamma\}$ for some singular positive semi-definite matrix Q, and $\gamma > 0$. Let $L = \ker(Q)$, which by Lemma 2 is the recession cone of $\mathcal{E}$. Suppose first that $e_n \in L$ and consider $H = \{x \in \mathbb{R}^n \mid x_n = 0\}$. Then for any $\bar{x} \in \mathcal{E}$, we have $\bar{x} - \bar{x}_n e_n \in \mathcal{E} \cap H$, and H has the desired property.

Thus, we may assume that $e_n \notin L$. We now apply Lemma 2, and obtain a decomposition

$$\mathcal{E} = \tilde{\mathcal{E}}|_{x_{i_1} = \bar{x}_{i_1}, \ldots, x_{i_k} = \bar{x}_{i_k}} + L.$$

Further, $\mathcal{E}' := \mathcal{E}|_{x_{i_1} = \bar{x}_{i_1}, \ldots, x_{i_k}}$ is an ellipsoid in $\mathbb{R}^{n-k}$. We note that since $e_n \notin \text{lin}(H)$, by the remark following Lemma 2, we may assume x_n is not among the variables fixed. Thus, we may assume that e'_n, the restriction of e_n obtained by dropping the fixed components, is non-zero in $\mathbb{R}^{n-k}$. We can now apply the proof of the bounded case above to the ellipsoid $\mathcal{E}'$. That is, there exists a hyperplane $H' \subseteq \mathbb{R}^{n-k}$ such that for any $x' \in \mathcal{E}'$ there exists $\lambda' \in \mathbb{R}$ such that $x' + \lambda' e'_n \in \mathcal{E}' \cap H'$.

Let $\tilde{H}'$ be obtained from H' by considering it in the original space $\mathbb{R}^n$, i.e., we have $x \in \tilde{H}'$ if and only if $x_{i_1} = \bar{x}_{i_1}, \ldots, x_{i_k} = \bar{x}_{i=k}$ and the vector consisting of components of x not among these x_{i_j} is in H'. We claim that the hyperplane $H := \tilde{H}' + L$ satisfies the condition of the lemma. By construction, $e_n \notin \text{lin}(H)$. Now for any $x \in \mathcal{E}$, there exists $l \in L$ such that $x - l \in \tilde{\mathcal{E}}'$. Then for some $\lambda \in \mathbb{R}$ we have $x - l + \lambda e_n \in \tilde{\mathcal{E}}' \cap H$ and since $L \subseteq H$ we have $x + \lambda e_n \in \mathcal{E} \cap H$. $\square$

With these results in hand, we are now ready to proceed to the proof of Theorem 1.

3 Proof of Theorem 1

To prove sufficiency of the condition, assume that we are given a set

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C},$$

where $\mathcal{E}_i = \{x \in \mathbb{R}^n \mid (x - c_i)^\top Q_i (x - c_i) \leq \gamma_i\}$ are ellipsoidal regions, $\mathcal{P}_i = \{x \in \mathbb{R}^n \mid A_i x \leq b_i\}$ are polytopes, and $\mathcal{C} = \mathrm{cone}\{r_1, \ldots, r_t\} \subseteq \mathbb{R}^n$ is a polyhedral cone. For each ellipsoidal region $\mathcal{E}_i$, if $\gamma_i > 0$ we can normalize the right hand side of the inequality to 1. Else, $\mathcal{E}_i$ is either empty or an affine space and γ_i can be set to 1 at the cost of adding additional linear inequalities to the system $A_i x \leq b_i$. Thus, we may assume $\gamma_i = 1$ for all $i = 1, \ldots, k$.

We introduce new continuous variables $x_i \in \mathbb{R}^n$ and binary variables $\delta_i \in \{0, 1\}$, for $i = 1, \ldots, k$, that will model the individual regions $\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}$. Then S can be described as the set of $x \in \mathbb{R}^n$ such that

$$\begin{aligned}
& x = \sum_{i=1}^{k} (x_i + \delta_i c_i) + \sum_{j=1}^{t} \lambda_j r_j \\
& A_i x_i \leq \delta_i (b_i - A_i c_i) && i = 1, \ldots, k \\
& \sum_{i=1}^{k} \delta_i = 1 \\
& \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}^\top \begin{pmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \leq 1 \\
& x_i \in \mathbb{R}^n, \ \delta_i \in \{0, 1\} && i = 1, \ldots, k \\
& \lambda_j \in \mathbb{R}^{\geq 0} && j = 1, \ldots, t.
\end{aligned}$$

Now if $\delta_1 = 1$ the remaining δ_i must be 0. Then for each x_i with $i \neq 1$, we have the constraint $A_i x_i \leq 0$ which has the single feasible point $x_i = 0$ since $\mathcal{P}_i$ is a polytope. The remaining constraints reduce to

$$\begin{aligned}
& x = x_1 + c_1 + \sum_{j=1}^{t} \lambda_j r_j \\
& A_1 (x_1 + c_1) \leq b_1 \\
& x_1^\top Q_1 x_1 \leq 1 \\
& x_1 \in \mathbb{R}^n \\
& \lambda_j \in \mathbb{R}^{\geq 0} && j = 1, \ldots, t.
\end{aligned}$$

By employing a change of variables $x' = x_1 + c_1$, it can be checked that the latter system describes the region $\mathcal{E}_1 \cap \mathcal{P}_1 + \mathcal{C}$. The remaining regions follow symmetrically. Therefore S is EMB-representable.

The remainder of the proof is devoted to proving necessity of the condition. We are given an ellipsoidal region $\mathcal{E}$ and a polyhedron $\mathcal{P}$ in $\mathbb{R}^{n+p+q}$, and we define

$$\bar{S} := \mathcal{E} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0,1\}^q),$$
$$S := \operatorname{proj}_n(\bar{S}).$$

We must show the existence of ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}.$$

Claim 1. *It suffices to find ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, and polyhedral cones $\mathcal{C}_i \subseteq \mathbb{R}^n$, for $i = 1, \dots, k$, that satisfy*

$$S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i). \tag{3}$$

Proof of claim. Let $\tilde{S} := \mathcal{E} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times [0,1]^q)$. Then for every $\bar{z} \in \mathbb{R}^q$, define $\bar{S}_{\bar{z}} := \mathcal{E} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{\bar{z}\})$. Clearly, for every $\bar{z} \in \{0,1\}^q$, we have $\operatorname{rec}(\bar{S}_{\bar{z}}) = \operatorname{rec}(\tilde{S})$, and so $\operatorname{proj}_n(\operatorname{rec}(\bar{S}_{\bar{z}})) = \operatorname{proj}_n(\operatorname{rec}(\tilde{S}))$. Since projections and recession cones operators commute for closed convex sets, we obtain $\operatorname{rec}(\operatorname{proj}_n(\bar{S}_{\bar{z}})) = \operatorname{proj}_n(\operatorname{rec}(\tilde{S}))$. Let $\mathcal{C} := \operatorname{proj}_n(\operatorname{rec}(\tilde{S}))$. By Observation 2, the set $\operatorname{rec}(\tilde{S})$ is a polyhedral cone, thus so is its projection $\mathcal{C}$.

Note that $\bar{S} = \cup_{\bar{z} \in \{0,1\}^q} \bar{S}_{\bar{z}}$ implies $S = \cup_{\bar{z} \in \{0,1\}^q} \operatorname{proj}_n(\bar{S}_{\bar{z}})$, therefore $\operatorname{rec}(S) = \mathcal{C}$. This concludes the proof since $S = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i) = \bigcup_{i=1}^{k} (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}$. $\diamond$

Claim 2. *It suffices to find ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$, and polyhedral cones $\mathcal{C}_i \subseteq \mathbb{R}^n$, for $i = 1, \dots, k$, that satisfy* (3).

Proof of claim. In order to show the claim, we prove that if we have $\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i$ for an ellipsoidal region $\mathcal{E}_i$, a polyhedron $\mathcal{P}_i$, and a polyhedral cone $\mathcal{C}_i$, then we may replace $\mathcal{P}_i$ with a polytope $\mathcal{R}$ without loss.

Replacing $\mathcal{C}_i$ with $\mathcal{C}_i + \operatorname{rec}(\mathcal{E}_i \cap \mathcal{P}_i)$ if necessary, we may assume that $\operatorname{rec}(\mathcal{E}_i \cap \mathcal{P}_i) \subseteq \mathcal{C}_i$. Note that the newly defined $\mathcal{C}_i$ is a polyhedral cone by Observation 2. Consider a polyhedral approximation $\mathcal{B}$ of $\mathcal{E}_i$ such that $\mathcal{B} \subseteq \mathbb{R}^n$ is a polyhedron, $\mathcal{E}_i \subseteq \mathcal{B}$, and $\operatorname{rec}(\mathcal{E}_i) = \operatorname{rec}(\mathcal{B})$. Then $\mathcal{B} \cap \mathcal{P}_i$ is a polyhedron and can be decomposed as $\mathcal{R} + \mathcal{C}'_i$ for a polytope $\mathcal{R}$ and a polyhedral cone $\mathcal{C}'_i \subseteq \mathcal{C}_i$. We claim that $\mathcal{E}_i \cap \mathcal{R} + \mathcal{C}_i = \mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i$.

Let $x \in \mathcal{E}_i \cap \mathcal{R} + \mathcal{C}_i$, and note that $\mathcal{R} \subseteq \mathcal{P}_i$ so that $x \in \mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i$. Thus, $\mathcal{E}_i \cap \mathcal{R} + \mathcal{C}_i \subseteq \mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i$. Let $x \in \mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i$. Then $x \in \mathcal{B} \cap \mathcal{P}_i + \mathcal{C}_i = \mathcal{R} + \mathcal{C}_i$ and we may write $x = r + c$ for some $r \in \mathcal{R}$, $c \in \mathcal{C}_i$. Note that $c \in \operatorname{rec}(\mathcal{E}_i)$, and since $\operatorname{rec}(\mathcal{E}_i)$ is a linear space by Lemma 2, we obtain $-c \in \operatorname{rec}(\mathcal{E}_i)$ as well. Then $x = (x - c) + c$ and $x - c = r \in \mathcal{E}_i \cap \mathcal{R}$, $c \in \mathcal{C}_i$ so $x \in \mathcal{E}_i \cap \mathcal{R} + \mathcal{C}_i$. $\diamond$

Claim 3. *We can assume without loss of generality $q = 0$.*

Proof of claim. Note that, using restrictions, we can write the set S in the form

$$S = \bigcup_{\bar{z} \in \{0,1\}^q} \operatorname{proj}_n(\bar{S}|_{z=\bar{z}}).$$

It suffices to show that each restriction $\bar{S}|_{z=\bar{z}} = \mathcal{E}' \cap \mathcal{P}'$ for some ellipsoidal region $\mathcal{E}' \subset \mathbb{R}^{n+p}$ and polyhedron $\mathcal{P}' \subseteq \mathbb{R}^{n+p}$. Then, assuming the result in the case $q = 0$, for each $\bar{z} \in \{0,1\}^q$ we have $\operatorname{proj}_n(\bar{S}|_{z=\bar{z}}) = \cup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i)$. Since S is the finite union of such sets, the result follows.

Let $\bar{z} \in \{0,1\}^q$. We note $\bar{S}|_{z=\bar{z}} = \mathcal{E}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}}$. By Lemma 1, $\mathcal{E}' := \mathcal{E}|_{z=\bar{z}}$ is an ellipsoidal region in $\mathbb{R}^{n+p}$. Let $\mathcal{P} = \{(x,y,z) \in \mathbb{R}^{n+p} \times \{0,1\}^q \mid Ax+By+Cz \leq d\}$. Also, $\mathcal{P}' := \mathcal{P}|_{z=\bar{z}} = \{(x,y) \in \mathbb{R}^{n+p} \mid Ax + By \leq d - C\bar{z}\}$ is clearly a polyhedron. ⋄

Claim 4. *We can assume without loss of generality $p = 1$.*

Proof of claim. Let $\mathcal{E} \cap \mathcal{P} \subseteq \mathbb{R}^{n+p}$. We prove that $S = \operatorname{proj}_n(\mathcal{E} \cap \mathcal{P})$ has the desired decomposition, by induction on p. For this claim, we assume the base case, $p = 1$. Now let $p = k$, and suppose the statement holds for $p < k$. Given $\mathcal{E} \cap \mathcal{P} \subseteq \mathbb{R}^{n+k}$, by the base case $p = 1$ we have

$$\operatorname{proj}_{n+k-1}(\mathcal{E} \cap \mathcal{P}) = \bigcup_{i=1}^{t} (\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i).$$

Since the projection of a union is the union of the projections, we have

$$S = \operatorname{proj}_n(\mathcal{E} \cap \mathcal{P}) = \bigcup_{i=1}^{t} \operatorname{proj}_n(\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i).$$

Now proj_n is a linear operator and by the induction hypothesis we have

$$S = \bigcup_{i=1}^{t} \Big(\bigcup_{j=1}^{s_i} (\mathcal{E}_{i,j} \cap \mathcal{P}_{i,j} + \mathcal{K}_{i,j}) + \mathcal{C}'_i \Big),$$

where $\mathcal{C}'_i := \operatorname{proj}_n(\mathcal{C}_i)$. Setting $\mathcal{K}'_{i,j} = \mathcal{K}_{i,j} + \mathcal{C}'_i$ for each $i = 1, \dots, t$ and $j = 1, \dots, s_i$, we have

$$S = \bigcup_{i=1}^{t} \Big(\bigcup_{j=1}^{s_i} (\mathcal{E}_{i,j} \cap \mathcal{P}_{i,j} + \mathcal{K}'_{i,j}) \Big),$$

and we are done. ⋄

To prove Theorem 1 it remains to show the following. Assume we are given $\mathcal{E} \cap \mathcal{P} \subseteq \mathbb{R}^{n+1}$. We must show the existence of ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$, and polyhedral cones $\mathcal{C}_i \subseteq \mathbb{R}^n$, for $i = 1, \dots, k$, that satisfy (3).

Given a half-space $H^+ = \{x \in \mathbb{R}^n \mid a^\top x \geq b\}$, we write H for the hyperplane $\{x \in \mathbb{R}^n \mid a^\top x = b\}$ and H^- for the half-space $\{x \in \mathbb{R}^n \mid a^\top x \leq b\}$. A polyhedron is the intersection of finitely many half-spaces. Thus, there exist half-spaces $H_1^+, \dots, H_s^+ \subseteq \mathbb{R}^{n+1}$ such that $\mathcal{P} = \cap_{i=1}^s H_i^+$. By Lemma 3, there exists a hyperplane $H_0 \subset \mathbb{R}^{n+1}$ with $e_{n+1} \notin \operatorname{lin}(H_0)$ such that $\operatorname{proj}_n(\mathcal{E}) = \operatorname{proj}_n(\mathcal{E} \cap H_0)$. Then

$$\mathcal{E} \cap \mathcal{P} = (\mathcal{E} \cap H_0^+ \cap_{i=1}^s H_i^+) \cup (\mathcal{E} \cap H_0^- \cap_{i=1}^s H_i^+),$$

and it suffices to show the statement for the region $\mathcal{E} \cap H_0^+ \cap_{i=1}^s H_i^+$.

Claim 5. *Let $\mathcal{H}$ be the collection of hyperplanes H among $H_0, \dots, H_s$ with $e_{n+1} \notin \operatorname{lin}(H)$. Then*

$$\operatorname{proj}_n(\mathcal{E} \cap_{i=0}^s H_i^+) = \bigcup_{H \in \mathcal{H}} \operatorname{proj}_n(\mathcal{E} \cap H \cap_{i=0}^s H_i^+).$$

Proof of claim. It suffices to show that $\mathcal{E} \cap_{i=0}^s H_i^+$ has the following property: for any $x \in \mathcal{E} \cap_{i=0}^s H_i^+$ there exists a hyperplane $H \in \mathcal{H}$ and a $\lambda \in \mathbb{R}$ such that $x + \lambda e_{n+1} \in \mathcal{E} \cap H \cap_{i=0}^s H_i^+$.

To prove the claim, we show that we can translate a point $x \in \mathcal{E} \cap_{i=0}^s H_i^+$ along the line $\{x + te_{n+1} \mid t \in \mathbb{R}\}$, and inside the feasible region, until it meets a half-space in $\mathcal{H}$ at equality. If $e_{n+1} \in \operatorname{lin}(H_i)$ for a half-space H_i, then $x + \lambda e_{n+1} \in H_i^+$ for any $\lambda \in \mathbb{R}$.

Let $\bar{x} \in \mathcal{E} \cap_{i=0}^s H_i^+$. Then, by the existence of the shadowing hyperplane H_0, there is one direction among $\{\pm e_{n+1}\}$ along which $\bar{x}$ may be translated to intersect H_0 while staying inside $\mathcal{E}$. Thus, there exists $\bar{\lambda} \in \mathbb{R}$ such that $\bar{x} + \bar{\lambda} e_{n+1} \in \mathcal{E} \cap_{i=0}^s H_i^+$ and $\bar{x} + \bar{\lambda} e_{n+1}$ is contained in at least one hyperplane $H \in \mathcal{H}$. $\diamond$

Then for any $H \in \mathcal{H}$ it suffices to show that there exists an ellipsoidal region $\mathcal{E}' \subseteq \mathbb{R}^n$ and a polyhedron $\mathcal{P}' \subseteq \mathbb{R}^n$ such that

$$\operatorname{proj}_n(\mathcal{E} \cap H \cap_{i=0}^s H_i^+) = \mathcal{E}' \cap \mathcal{P}'.$$

Without loss of generality, we may assume that $H_i \cap H \neq \emptyset$ for each $i = 0, \dots, s$. If not, say $H_j \cap H = \emptyset$ for some $0 \leq j \leq s$. Then either $\mathcal{E} \cap H \cap H_j^+ = \emptyset$ and our region is empty, or $\mathcal{E} \cap H \cap H_j^+ = \mathcal{E} \cap H$ and H_j^+ is redundant and may be removed.

We now show that each half-space H_i^+ can be replaced with a different half-space M_i^+ such that $\mathcal{E} \cap H \cap H_i^+ = \mathcal{E} \cap H \cap M_i^+$ and $e_{n+1} \in \operatorname{lin}(M_i^+)$. Without loss of generality, consider H_1^+ and the region $\mathcal{E} \cap H \cap H_1^+$. Let $U = H \cap H_1$. Then U is an $(n-1)$-dimensional affine space, say $U = v + V$ for a linear space V of dimension $n-1$. Let $W = V + \operatorname{span}(e_{n+1})$. Since $e_{n+1} \notin \operatorname{lin}(U)$, $M_1 := v + W$ is a hyperplane in $\mathbb{R}^{n+1}$ that divides H into the same two regions that H_1 does. In particular, upon choice of direction, we have that M_1^+ has the desired properties. We may now replace each H_i^+ with M_i^+ in this way.

By the requirement $e_{n+1} \in \operatorname{lin}(M_i^+)$, we have that each M_i^+ is defined by a linear inequality with the coefficient of x_{n+1} equal to 0. Thus, the projection

$\text{proj}_n(M_i^+)$ is a half-space in $\mathbb{R}^n$ which we denote $\bar{H}_i^+$. Further, if each H_i^+ for $i = 0, \ldots, s$ is replaced in this way, we have

$$\text{proj}_n(\mathcal{E} \cap H \cap_{i=0}^s H_i^+) = \text{proj}_n(\mathcal{E} \cap H \cap_{i=0}^s M_i^+) = \text{proj}_n(\mathcal{E} \cap H) \cap_{i=0}^s \bar{H}_i^+,$$

and we have the desired polyhedron $\mathcal{P}' := \cap_{i=0}^s \bar{H}_i^+$.

It remains to show that $\text{proj}_n(\mathcal{E} \cap H)$ is an ellipsoidal region $\mathcal{E}' \subseteq \mathbb{R}^n$. Let $H = \{(x, y) \in \mathbb{R}^{n+1} \mid a^\top(x, y) = b\}$. Then there exists a linear transformation from $\mathbb{R}^{n+1}$ to itself, defined by the matrix A whose first n rows are the first n standard unit vectors of $\mathbb{R}^{n+1}$ and whose last row is a. Moreover, A is invertible since e_{n+1} is not in $\text{lin}(H)$. Then, by construction of A, for any vector $(x, y) \in \mathbb{R}^{n+1}$ we have $A(x, y) = (x, c)$ for some $c \in \mathbb{R}$. Furthermore, $A(H)$ gets mapped to the hyperplane $\{(x, y) \in \mathbb{R}^{n+1} \mid y = b\}$. Now, since A is invertible we have

$$\begin{aligned} x \in \text{proj}_n(\mathcal{E} \cap H) &\Leftrightarrow \exists y \in \mathbb{R} \text{ such that } (x, y) \in \mathcal{E} \cap H \\ &\Leftrightarrow (x, b) \in A(\mathcal{E} \cap H) \\ &\Leftrightarrow (x, b) \in A(\mathcal{E}). \end{aligned}$$

This shows that $\text{proj}_n(\mathcal{E} \cap H) = A(\mathcal{E})|_{y=b}$. Ellipsoidal regions are clearly preserved under invertible linear transformations, therefore $A(\mathcal{E})$ is an ellipsoidal region. Finally, by Lemma 1, the set $A(\mathcal{E})|_{y=b}$ is an ellipsoidal region. This concludes the proof that $\text{proj}_n(\mathcal{E} \cap H)$ is an ellipsoidal region $\mathcal{E}'$. □

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